USPAS2009 COURSE ON COLLECTIVE EFFECTS IN BEAM DYNAMICS FOR PARTICLE ACCELERATORS

Elias Métral and Giovanni Rumolo

- Introduction (EM) & Introduction (GR)
- Space charge (EM)
- Envelope equations (EM)
- Wake fields and impedances (EM)
- Longitudinal beam dynamics (GR)
- Transverse beam dynamics (EM)
- Two-stream effects (GR)
- Numerical modeling (GR)
- HEADTAIL code (GR)

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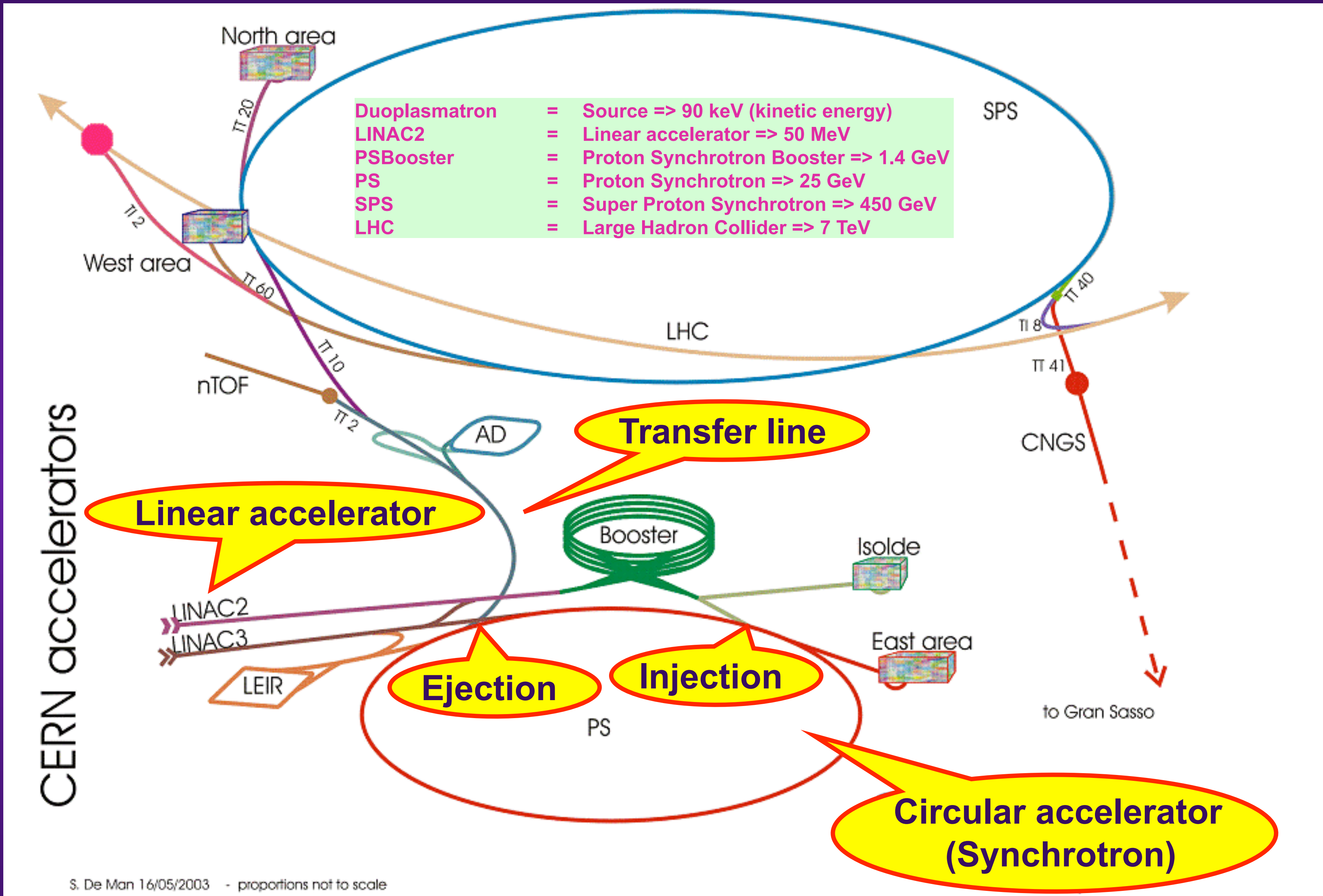
PROGRAM OF THE WEEK

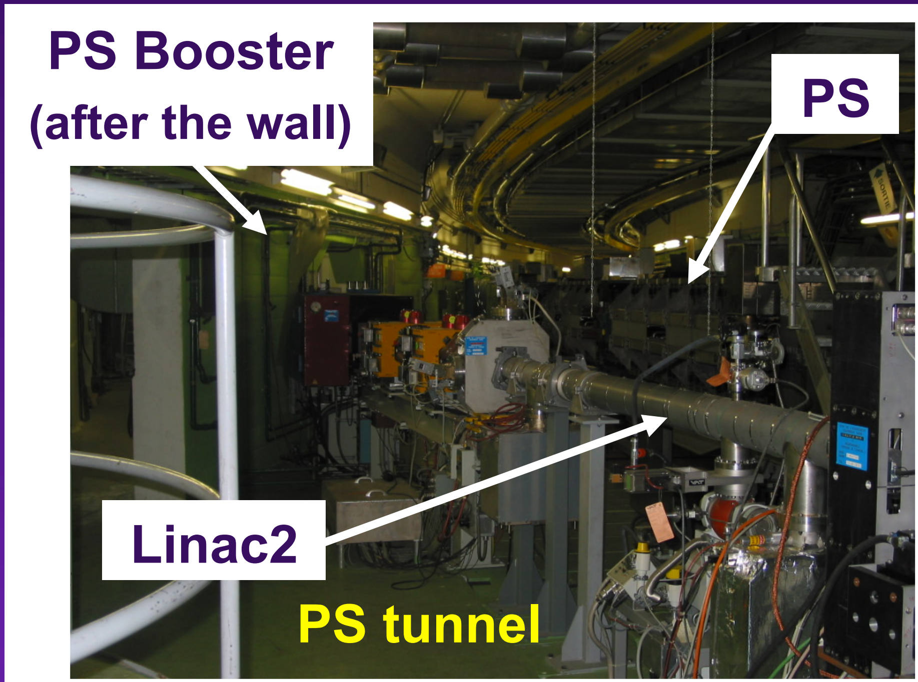
	09:00 – 09:55	10:00 – 10:55	11:00 – 11:55	12:00 – 12:55	14:30 – 15:25	15:30 – 17:00
MO 22/06/09 (EM)	Introduction (EM & GR)	Space charge	Envelope equations	Wake fields & impedances	Wake fields & impedances	Tutorials
TU 23/06/09 (GR)	Correction of tutorials	Longitudinal dynamics	Longitudinal dynamics	Longitudinal dynamics	Longitudinal dynamics	Tutorials
WE 24/06/09 (EM)	Correction of tutorials	Transverse dynamics (Coasting)	Transverse dynamics (Coasting)	Transverse dynamics (Bunched)	Transverse dynamics (Bunched)	Tutorials
TH 25/06/09 (GR)	Correction of tutorials	Two-stream effects	Two-stream effects	Numerical modeling	HEADTAIL code	Tutorials
FR 26/06/09	Exam	Exam	Exam			

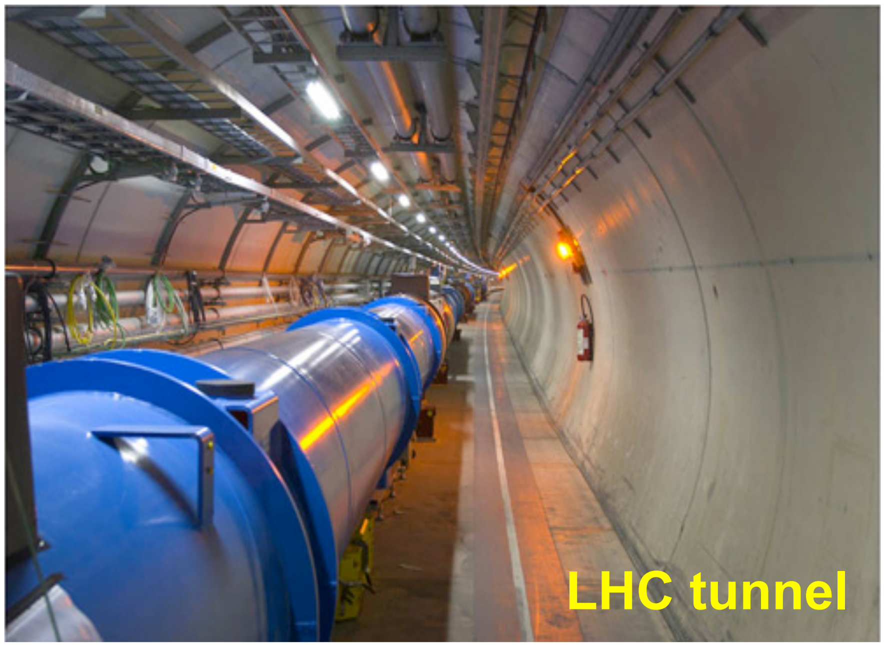
INTRODUCTION (2/35)



Example of the LHC p beam in the injector chain

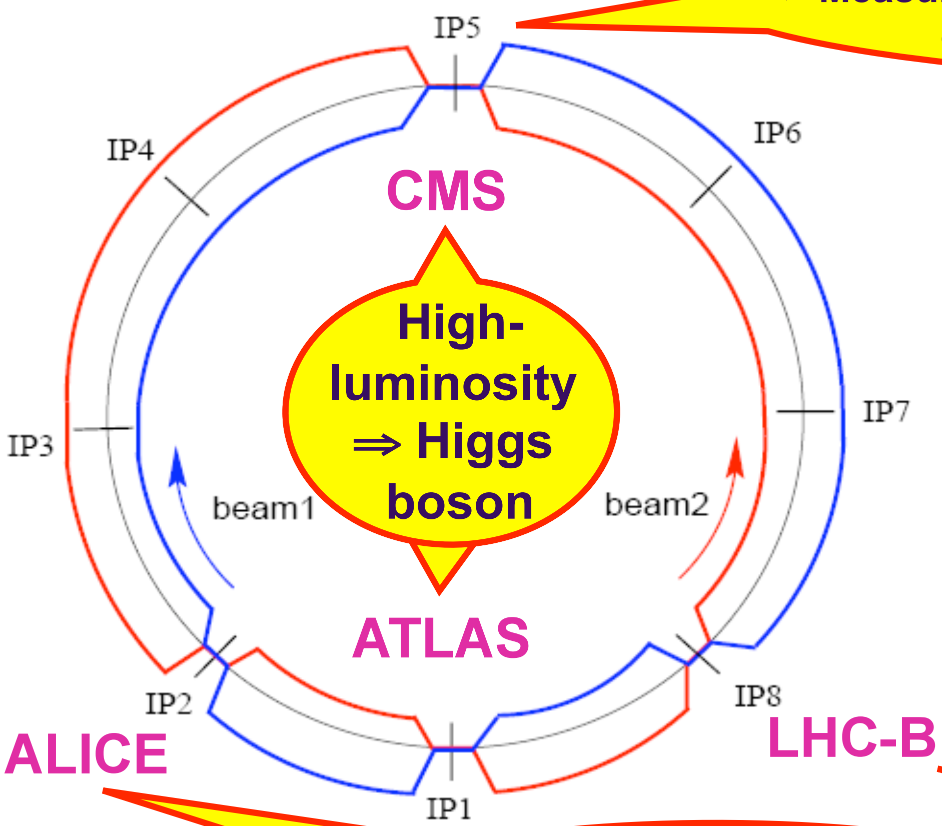
INTRODUCTION (3/35)





LAYOUT OF THE LHC

Courtesy W. Herr



High-luminosity ⇒ Higgs boson

+ TOTEM

⇒ Measure the total proton-proton cross-section and study elastic scattering and diffractive physics

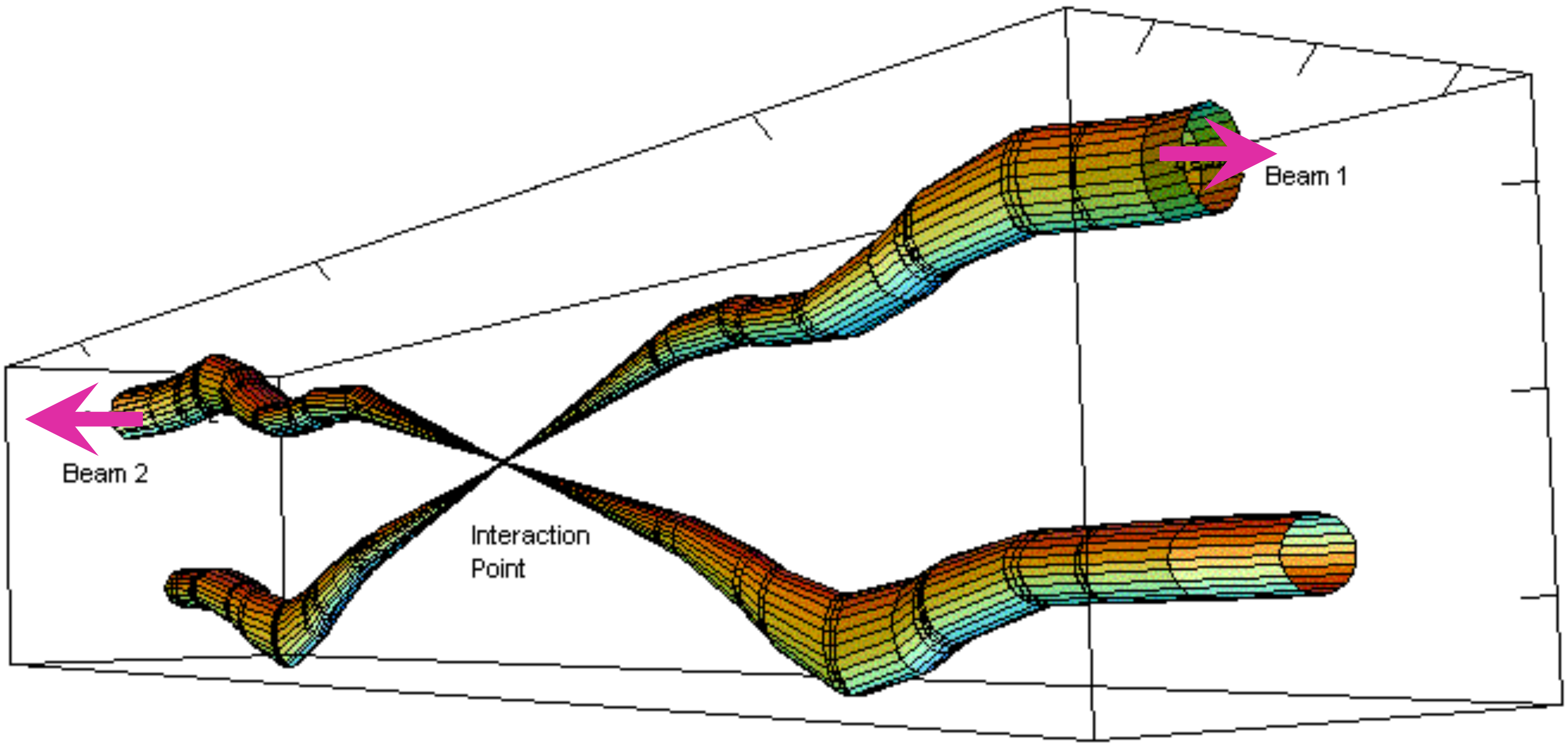
IP = Interaction Point

Ions ⇒ New phase of matter expected: Quark-Gluon Plasma (QGP)

Beauty quark physics

INTRODUCTION (5/35)

COLLISION in IP1 (ATLAS)



Relative beam sizes around IP1 (Atlas) in collision

⇒ Vertical crossing angle in IP1 (ATLAS) and horizontal one in IP5 (CMS)

FIGURE OF MERIT for a synchrotron / collider: Brightness / luminosity

- **(2D) BEAM BRIGHTNESS**

$$B = \frac{I}{\pi^2 \, \varepsilon_x \, \varepsilon_y}$$

Beam current (I)

Transverse emittances (ε_x, ε_y)

- **MACHINE LUMINOSITY**

$$L = \frac{N_{events/second}}{\sigma_{event}}$$

L: [cm^{-2} s^{-1}]

$N_{events/second}$: Number of events per second generated in the collisions

σ_{event}: Cross-section for the event under study

- **The Luminosity depends only on the beam parameters**
  ⇒ It is independent of the physical reaction

- **Reliable procedures to compute and measure**

⇒ **For a Gaussian (round) beam distribution**

$$L = \frac{N_b^2 \, M \, f_0 \, \gamma}{4 \, \pi \, \varepsilon_n \, \beta^*} F_{ca}$$

- Number of particles per bunch (N_b)
- Number of bunches per beam (M)
- Revolution frequency (f_0)
- Relativistic mass factor (γ)
- Normalized rms transverse beam emittance (ε_n)
- β-function at the collision point (β^*)
- Geometric reduction factor due to the crossing angle at the IP (F_{ca})

◆ **PEAK LUMINOSITY for ATLAS&CMS in the LHC =** $L_{peak} = 10^{34} \text{ cm}^{-2} \text{ s}^{-1}$

INTRODUCTION (8/35)

Number of particles per bunch	N_b	1.15×10^{11}
Number of bunches per beam	M	2808
Revolution frequency	f_0	11245 Hz
Relativistic velocity factor	γ	7461 (=> E = 7 TeV)
β-function at the collision point	β^*	55 cm
Normalised rms transverse beam emittance	ε_n	3.75×10^{-4} cm
Geometric reduction factor	F_{ca}	0.84

$$F_{ca} = 1 / \sqrt{1 + \left(\frac{\theta_c \, \sigma_s}{2 \, \sigma^*} \right)^2}$$

Full crossing angle at the IP	θ_c	285 μrad
Rms bunch length	σ_s	7.55 cm
Transverse rms beam size at the IP	σ^*	16.7 μm

- **INTEGRATED LUMINOSITY** $L_{\text{int}} = \int_0^T L(t)\, dt$

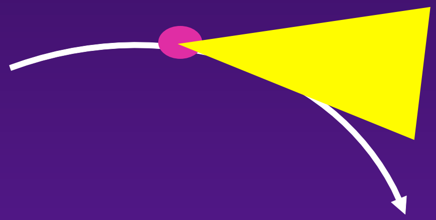
  $\Rightarrow$ **The real figure of merit =** $L_{\text{int}}\, \sigma_{event} = \text{number of events}$

- **LHC integrated Luminosity expected per year: [80-120] fb^{-1}**

Reminder: 1 barn = 10^{-24} cm^2
and femto = 10^{-15}

INTRODUCTION (10/35)

SYNCHROTRON RADIATION



- **Power radiated by a particle (due to bending)**

$$P_{\perp} = \frac{q^2 \, c \, \beta^4 \, E_{total}^4}{6 \, \pi \, \varepsilon_0 \, \rho_{curv}^2 \, E_{rest}^4}$$

Relativistic velocity factor

Particle total energy

Curvature radius of the dipoles

Particle rest energy

- **Energy radiated in one ring revolution**

$$U_0 = \frac{q^2 \, \beta^3 \, E_{total}^4}{3 \, \varepsilon_0 \, E_{rest}^4 \, \rho_{curv}}$$

- **Average (over the ring circumference) power radiation**

$$P_{av} = \frac{U_0}{T_0}$$

Revolution period

INTRODUCTION (11/35)

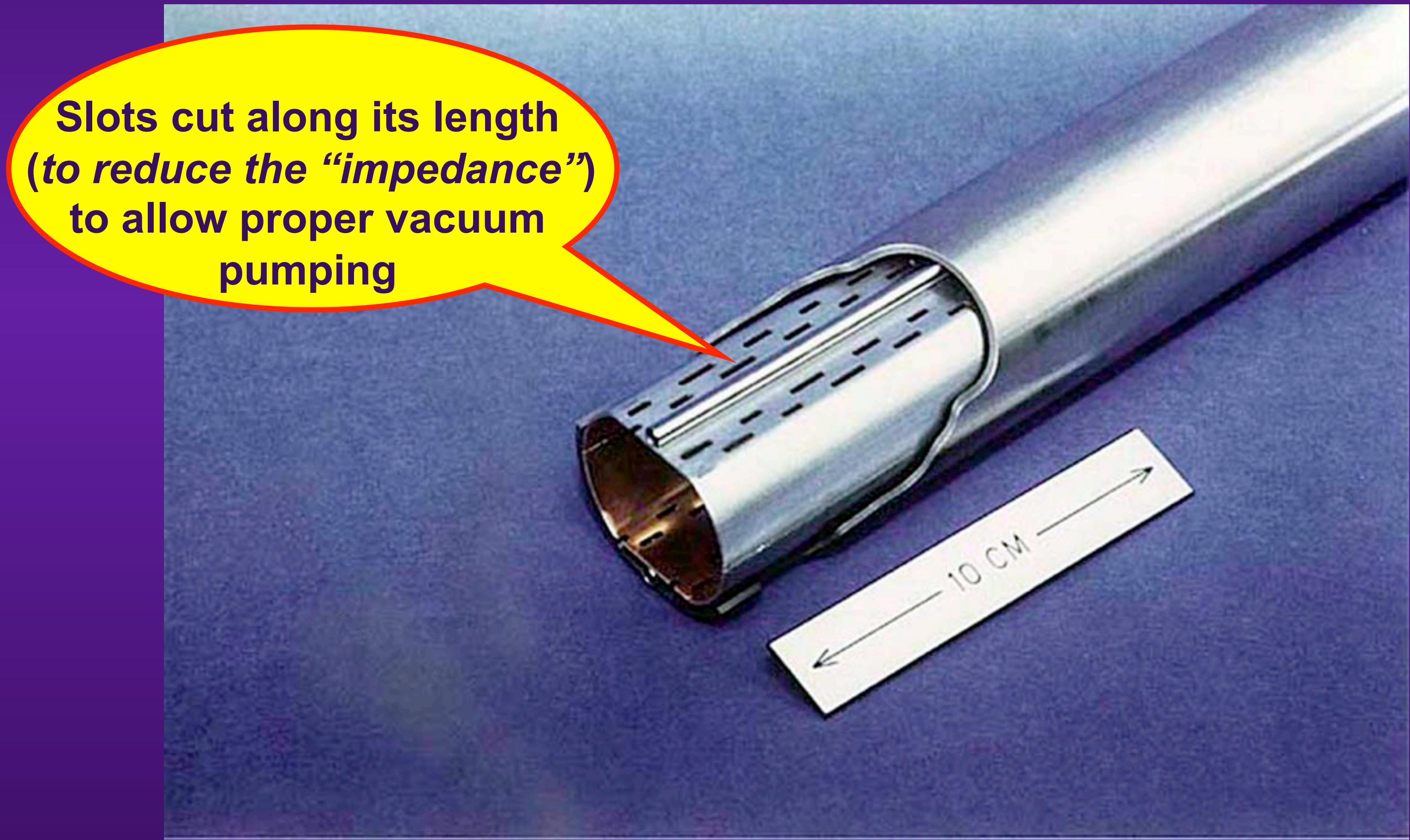
	LEP	LHC
ρ_{curv} [m]	3096.175	2803.95
p_0 [GeV/c]	104	7000
U_0	3.3 GeV	6.7 keV

The RF system had therefore to compensate for an energy lost of ~3% of the total beam energy per turn!

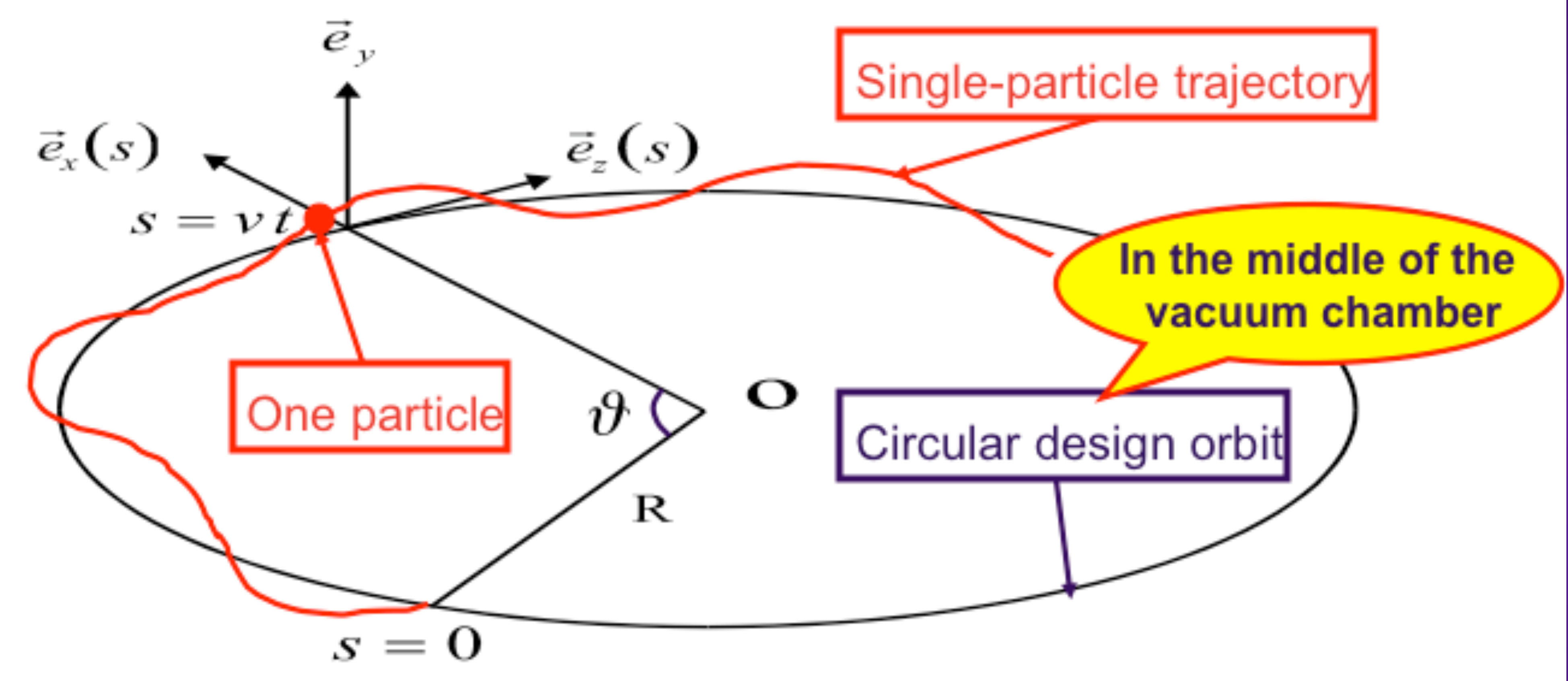
The total average (over the ring circumference) power radiation (per beam) is 3.9 kW (2808 bunches of 1.15 10^{11} protons)

INTRODUCTION (12/35)

- **LHC is the 1st proton storage ring for which synchrotron radiation becomes a noticeable effect => It gives rise to a significant heat load at top energy, which is intercepted by a beam screen at an elevated temperature of 5-20 K**



ACCELERATOR MODEL



$$C = 2\,\pi\,R \qquad v = \beta\,c = R\,\Omega_0 \qquad \Omega_0 = 2\pi\,f_0 = \frac{2\pi}{T_0}$$

Sometimes υ will also be used in the course

INTRODUCTION (14/35)

- **Transverse equation of motion**

$$\frac{1}{2}\beta_x \beta_x'' - \frac{1}{4}\beta_x'^2 + K_x(s)\beta_x^2 = 1$$

$$\mu_x(s) = \int_0^s \frac{dt}{\beta_x(t)}$$

$$Q_{x0} = \frac{\mu_x(C)}{2\pi}$$

- **Smooth approximation** $\beta_x(s) = \text{Constant} = < \beta_x >$

=> $$< \beta_x > = \frac{1}{\sqrt{K_x}} = \frac{R}{Q_{x0}}$$

$$Q_{x0} = \frac{\omega_{x0}}{\Omega_0}$$

$$x' = p_x$$

$$p_x' = \frac{F_x}{\beta^2 E_{total}}$$

$$F_x = F_x^{ext} + F_x^{pert}$$

$$\frac{F_x^{ext}}{\beta^2 E_{total}} = -K_x(s)\, x$$

$$\frac{d^2 x}{ds^2} + \left(\frac{Q_{x0}}{R}\right)^2 x = \frac{F_x^{pert}}{\beta^2 E_{total}}$$

$$\frac{d^2 x}{dt^2} + \omega_{x0}^2\, x = \frac{F_x^{pert}}{\gamma\, m_0}$$

- **Longitudinal equation of motion**

$$V = \hat{V}_{\mathrm{RF}} \sin \phi_{\mathrm{RF}}(t)$$

$$\phi_{\mathrm{RF}}(t) = \omega_{\mathrm{RF}}\, t + \phi_s$$

$$Q_{s0} = \frac{\omega_{s0}}{\Omega_0}$$

$$Q_{s0} = \left(- \frac{e \hat{V}_{\mathrm{RF}}\, h\, \eta \cos \phi_s}{2 \pi \beta^2 E_{total}} \right)^{1/2}$$

$$\delta = \frac{\Delta p}{p}$$

$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$

$$z' = - \eta\, \delta$$

$$\delta' = \frac{F_z}{\beta^2 E_{total}}$$

$$F_z = F_z^{ext} + F_z^{pert}$$

$$\frac{F_z^{ext}}{\beta^2 E_{total}} = \frac{1}{\eta} K_z(s)\, z = \frac{1}{\eta} \left(\frac{Q_{s0}}{R} \right)^2 z$$

$$\frac{d^2 z}{ds^2} + \left(\frac{Q_{s0}}{R} \right)^2 z = - \eta \frac{F_z^{pert}}{\beta^2 E_{total}}$$

$$\frac{d^2 z}{dt^2} + \omega_{s0}^2\, z = - \eta \frac{F_z^{pert}}{\gamma\, m_0}$$

INTRODUCTION (16/35)

- **Phase space coordinates**

Hamiltonian

$$\dot{q}_{psc} = \frac{\partial H}{\partial p_{psc}}$$

$$\dot{p}_{psc} = -\frac{\partial H}{\partial q_{psc}}$$

Transverse

$$q_{psc} = x$$

$$p_{psc} = \frac{R}{Q_x} p_x$$

Longitudinal

$$q_{psc} = z$$

$$p_{psc} = -\frac{\eta C}{2\pi Q_s} \delta$$

- **Polar coordinates**

$$q_{psc} = r \cos\phi$$

$$p_{psc} = -r \sin\phi$$

Transverse

$$x = r_x \cos\phi_x$$

$$p_x = -\frac{Q_x}{R} r_x \sin\phi_x$$

$$\phi_x = \frac{2\pi Q_x}{C} s + \phi_{x0}$$

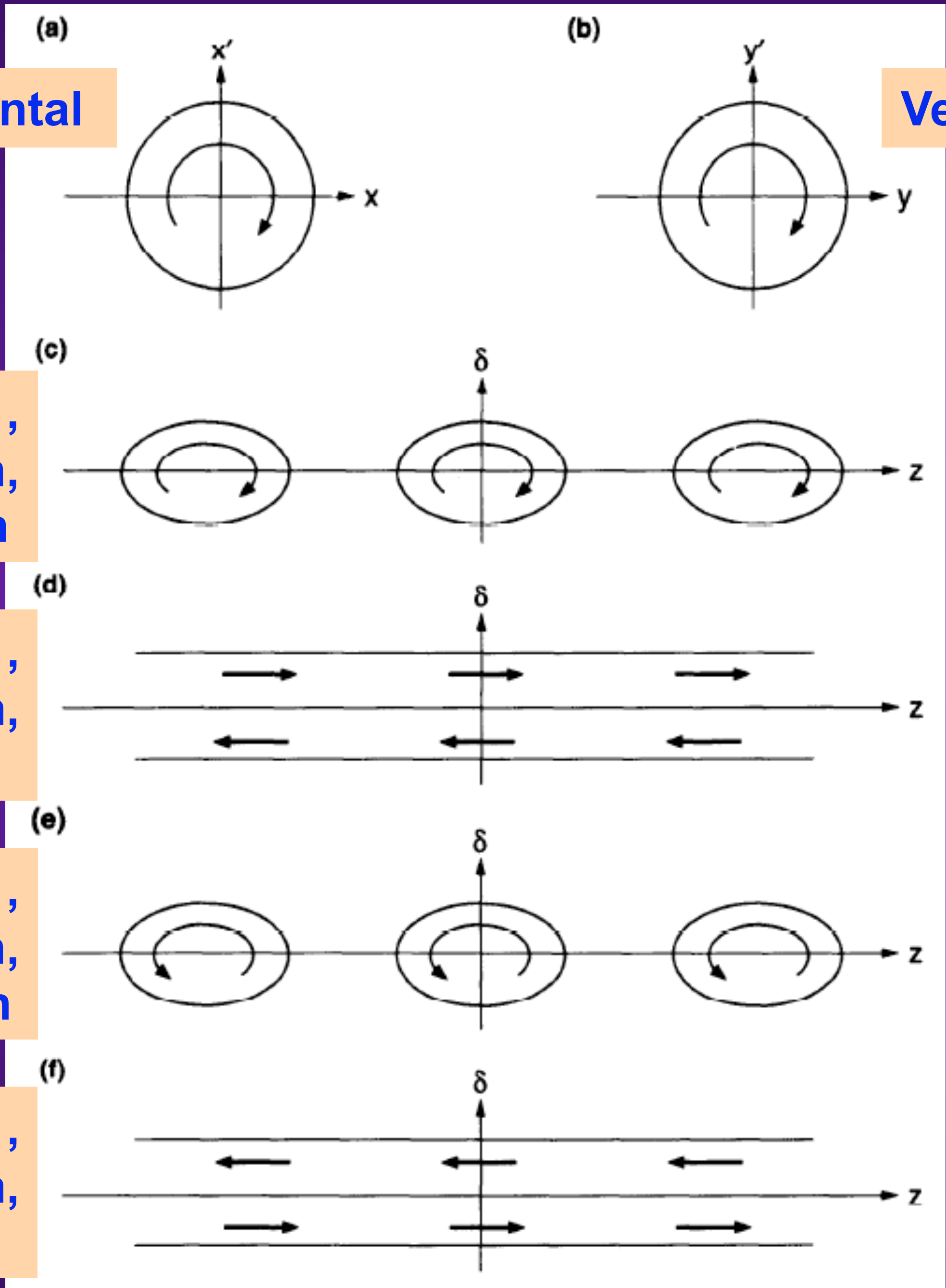
Longitudinal

$$z = r_z \cos\phi_z$$

$$\frac{\eta C}{2\pi Q_s} \delta = r_z \sin\phi_z$$

$$\phi_z = \frac{2\pi Q_s}{C} s + \phi_{z0}$$

INTRODUCTION (17/35)



Horizontal

Vertical

Longitudinal, bunched beam, below transition

Longitudinal, unbunched beam, below transition

Longitudinal, bunched beam, above transition

Longitudinal, unbunched beam, above transition

Courtesy of A.W. Chao

REMINDERS: (1) RELATIVISTIC EQUATIONS

$$E_{rest} = m_0 \, c^2$$

$$\gamma = \frac{E_{total}}{E_{rest}} = \frac{m}{m_0} = \frac{1}{\sqrt{1-\beta^2}}$$

$$\beta = \frac{v}{c}$$

$$\vec{p} = m \, \vec{v}$$

$$E_{total}^2 = E_{rest}^2 + p^2 \, c^2$$

For a particle of charge e

$$\frac{d\,\vec{p}}{d\,t} = \vec{F} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

(2) LORENTZ FORCE

$$\vec{F} = e \left(\vec{E} + \vec{v} \times \vec{B} \right)$$

- **Cartesian (x,y,s)**

$$F_x = e \left(E_x - v\, B_y \right)$$

$$F_y = e \left(E_y + v\, B_x \right)$$

$$F_s = e\, E_s$$

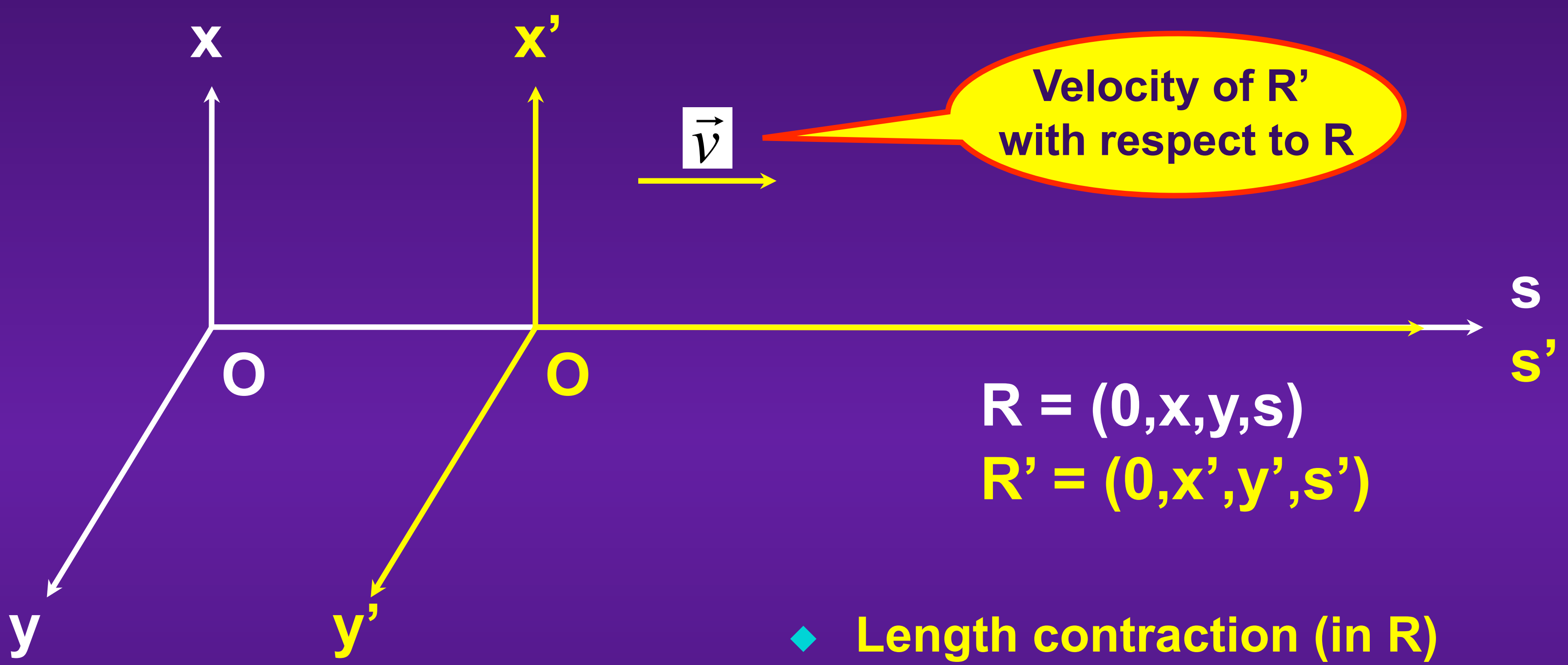
- **Cylindrical (r,θ,s)**

$$F_r = e \left(E_r - v\, B_\vartheta \right)$$

$$F_\vartheta = e \left(E_\vartheta + v\, B_r \right)$$

$$F_s = e\, E_s$$

(3) LORENTZ TRANSFORM



$$x = x' \qquad y = y'$$

$$s = \gamma \left(s' + v\, t' \right)$$

$$t = \gamma \left(\frac{v}{c^2} s' + t' \right)$$

- **Length contraction (in R)**

$$d s = \frac{d s'}{\gamma} \quad \text{for} \quad d t = 0$$

- **Time dilatation (in R)**

$$d t = \gamma\, d t' \quad \text{for} \quad d s' = 0$$

(4) MAXWELL EQUATIONS

- **Differential forms**

$$div\ \vec{E} = \frac{\rho}{\varepsilon}$$

Gauss's law for electric charge

$$div\ \vec{H} = 0$$

Gauss's law for magnetic charge

$$\overrightarrow{rot}\ \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

Faraday's and Lenz law

$$\overrightarrow{rot}\ \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

Ampere's law

- **Integral forms**

$$\iiint div\ \vec{E}\ dV = \iint \vec{E} . d\vec{S} = \frac{1}{\varepsilon} \iiint \rho\ dV$$

$$\iiint div\ \vec{H}\ dV = \iint \vec{H} . d\vec{S} = 0$$

$$\iint \overrightarrow{rot}\ \vec{E} . d\vec{S} = \oint \vec{E} . d\vec{s} = -\mu \iint \frac{\partial \vec{H}}{\partial t} . d\vec{S}$$

$$\iint \overrightarrow{rot}\ \vec{H} . d\vec{S} = \oint \vec{H} . d\vec{s} = \iint \vec{J} . d\vec{S} + \varepsilon \iint \frac{\partial \vec{E}}{\partial t} . d\vec{S}$$

with

$$\vec{B} = \mu\ \vec{H}$$

$$\vec{D} = \varepsilon\ \vec{E}$$

$$\vec{J} = \rho \vec{v} + \sigma \vec{E}$$

Maxwell equations valid in homogeneous, isotropic, continuous media

INTRODUCTION (22/35)

(5) NABLA, GRAD, ROT, DIV and LAPLACIAN OPERATORS

- **Cartesian (x,y,s)**

$$\vec{\nabla} \equiv \left| \begin{array}{c} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial s} \end{array} \right.$$

- **Cylindrical (r,θ,s)**

$$\vec{\nabla} \equiv \left| \begin{array}{c} \dfrac{\partial}{\partial r} \\ \dfrac{1}{r}\left(\dfrac{\partial}{\partial \vartheta}\right) \\ \dfrac{\partial}{\partial s} \end{array} \right.$$

Also noted $\overrightarrow{curl}\,\vec{E}$ or $\vec{\nabla} \wedge \vec{E}$

$$\overrightarrow{grad}\,\rho \equiv \vec{\nabla}\,\rho = \left| \begin{array}{c} \dfrac{\partial \rho}{\partial x} \\ \dfrac{\partial \rho}{\partial y} \\ \dfrac{\partial \rho}{\partial s} \end{array} \right.$$

$$\overrightarrow{rot}\,\vec{E} \equiv \vec{\nabla} \times \vec{E} = \left| \begin{array}{c} \dfrac{\partial E_s}{\partial y} - \dfrac{\partial E_y}{\partial s} \\ \dfrac{\partial E_x}{\partial s} - \dfrac{\partial E_s}{\partial x} \\ \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} \end{array} \right.$$

$$\overrightarrow{grad}\,\rho = \left| \begin{array}{c} \dfrac{\partial \rho}{\partial r} \\ \dfrac{1}{r}\left(\dfrac{\partial \rho}{\partial \vartheta}\right) \\ \dfrac{\partial \rho}{\partial s} \end{array} \right.$$

$$\overrightarrow{rot}\,\vec{E} = \left| \begin{array}{c} \dfrac{1}{r}\left(\dfrac{\partial E_s}{\partial \vartheta}\right) - \dfrac{\partial E_\theta}{\partial s} \\ \dfrac{\partial E_r}{\partial s} - \dfrac{\partial E_s}{\partial r} \\ \dfrac{1}{r}\left[\dfrac{\partial (r\,E_\theta)}{\partial r} - \dfrac{\partial E_r}{\partial \theta}\right] \end{array} \right.$$

$$div\,\vec{E} \equiv \vec{\nabla} . \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_s}{\partial s}$$

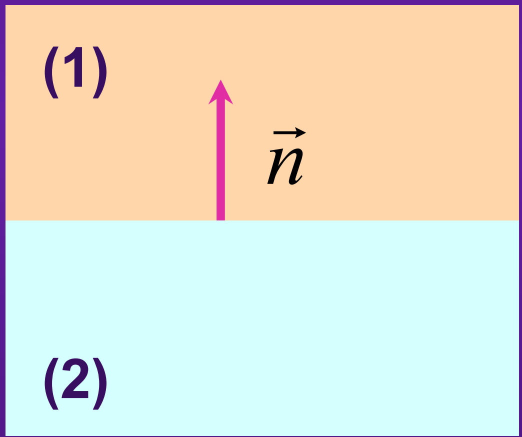
$$div\,\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,E_r\right) + \frac{1}{r}\frac{\partial E_\theta}{\partial \theta} + \frac{\partial E_s}{\partial s}$$

$$\Delta\,\rho \equiv \nabla^2 \rho = \text{Laplacian operator}$$
$$= \frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} + \frac{\partial^2 \rho}{\partial s^2}$$

$$\Delta\,\rho = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial \rho}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \rho}{\partial \theta^2} + \frac{\partial^2 \rho}{\partial s^2}$$

(6) GENERAL FIELD MATCHING CONDITIONS

Consider a surface separating two media "1" and "2". The following boundary conditions can be derived from Maxwell equations for the normal (⊥) and parallel (//) components of the fields at the surface

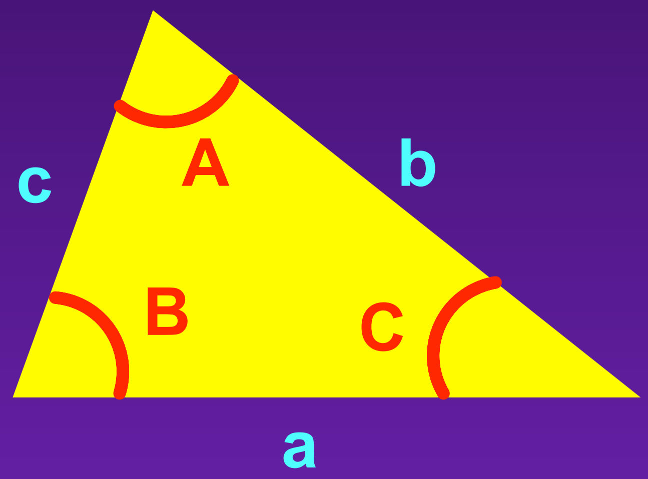


$$\vec{E}^1_{//} = \vec{E}^2_{//} \qquad \vec{H}^1_{//} - \vec{H}^2_{//} = \vec{K}$$

$$D^1_{\perp} - D^2_{\perp} = \Sigma \qquad B^1_{\perp} = B^2_{\perp}$$

where Σ is the surface charge density and $\vec{K}$ is the surface current density

(7) RELATIONS IN A TRIANGLE



$$a^2 = b^2 + c^2 - 2\,b\,c\cos \mathrm{A}$$

$$b^2 = a^2 + c^2 - 2\,a\,c\cos \mathrm{B}$$

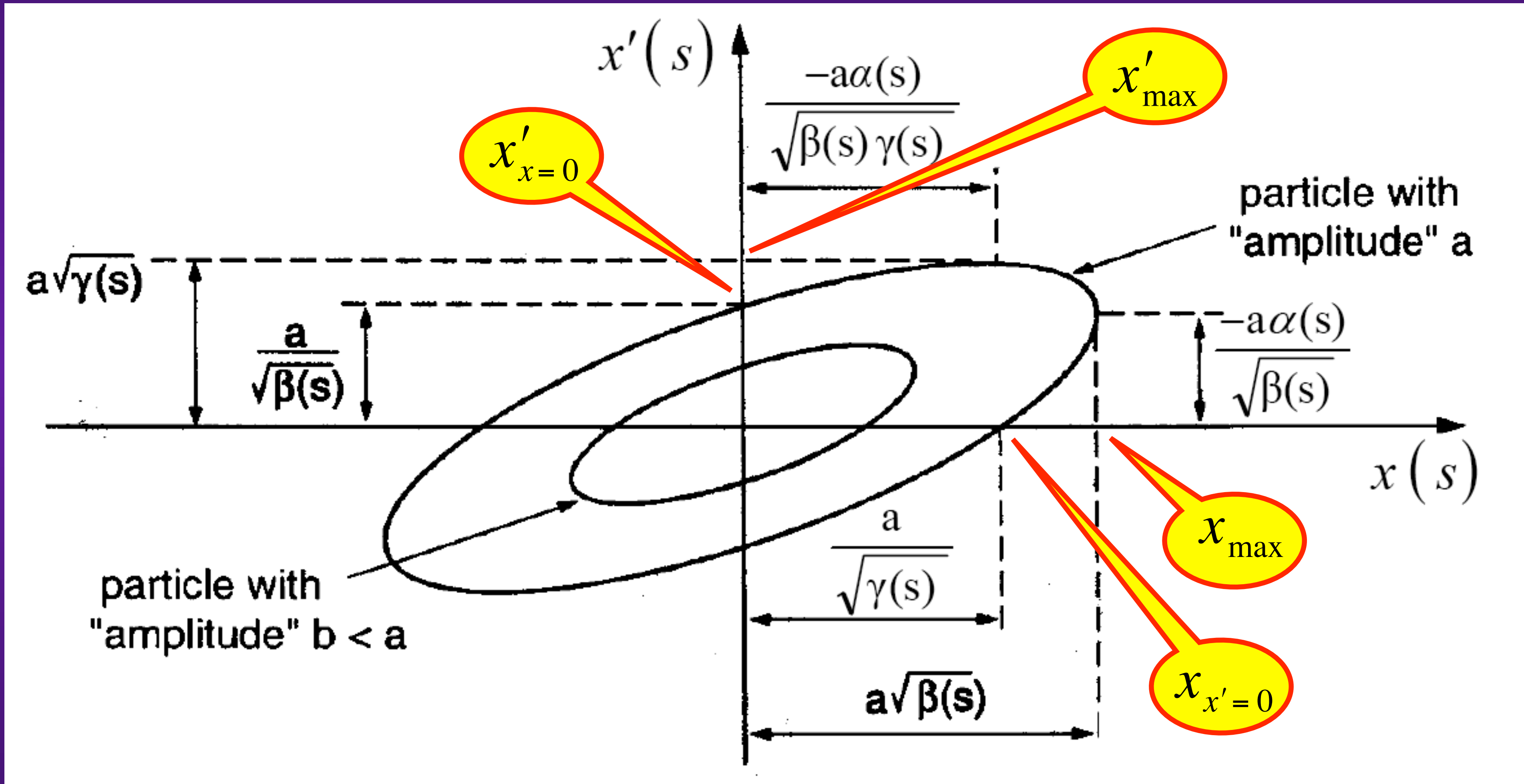
$$c^2 = a^2 + b^2 - 2\,a\,b\cos \mathrm{C}$$

Radius of the circumscribed circle

$$2\,R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$S = \frac{a\,b\sin C}{2} = \frac{a\,c\sin B}{2} = \frac{b\,c\sin A}{2}$$

(8) RELATIONS IN AN ELLIPSE: Example of the phase space ellipse in Transverse Beam dynamics



General relation in an ellipse

$$A = \pi \; x_{max} \; x'_{x=0} = \pi \; x_{x'=0} \; x'_{max}$$

INTRODUCTION (26/35) (9) USEFUL RELATIONS

- **Gaussian distribution** $$\lambda(s) = \frac{q}{\sqrt{2\pi}\,\sigma_s}\, e^{-\frac{s^2}{2\sigma_s^2}}$$

- $$x' = \frac{dx}{ds} \qquad \dot{x} = \frac{dx}{dt}$$

- **Parabolic distribution** $$\lambda(s) = \frac{3q}{2L}\left[1 - \left(\frac{2s}{L}\right)^2\right]$$

- $$\int_{s=0}^{\infty} \frac{ds}{\left(a^2+s\right)^2} = \frac{1}{a^2}$$

- $$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

- $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

- $$\int_{s=0}^{\infty} \frac{ds}{\left(a^2+s\right)^3} = \frac{1}{2a^4}$$

- $$\int_{s=0}^{\infty} \frac{ds}{\left(a^2+s\right)^{3/2}\left(b^2+s\right)^{1/2}} = \frac{2}{a(a+b)}$$

- $$\int_{s=0}^{\infty} \frac{ds}{\left(a^2+s\right)^4} = \frac{1}{3a^6}$$

- $$\int_{s=0}^{\infty} \frac{ds}{\left(a^2+s\right)^{5/2}\left(b^2+s\right)^{1/2}} = \frac{2(2a+b)}{3a^3(a+b)^2}$$

- $$\int_{s=0}^{\infty} \frac{ds}{\left(a^2+s\right)^5} = \frac{1}{4a^8}$$

INTRODUCTION (27/35)

- $$\int_{s=0}^{\infty} \frac{ds}{\left(a^2+s\right)^{3/2}\left(b^2+s\right)^{3/2}} = \frac{2}{ab\left(a+b\right)^2}$$

- $$\delta_{m0} = \begin{vmatrix} 1 & \text{if} & \text{m}=0 \\ 0 & \text{if} & \text{m}\neq 0 \end{vmatrix}$$

- $$\vec{\nabla}.\left(\vec{A}\times\vec{B}\right) = \vec{B}.\left(\vec{\nabla}\times\vec{A}\right) - \vec{A}.\left(\vec{\nabla}\times\vec{B}\right)$$

- $$curl\ curl = grad\ div - \Delta$$

- $$\vec{\nabla}\times\left(\vec{A}\times\vec{B}\right) = \vec{A}\left(\vec{\nabla}.\vec{B}\right) - \vec{B}.\left(\vec{\nabla}.\vec{A}\right) + \left(\vec{B}.\vec{\nabla}\right)\vec{A} - \left(\vec{A}.\vec{\nabla}\right)\vec{B}$$

- $$J_n'\left(y\right) = \frac{n}{y} J_n\left(y\right) - J_{n+1}\left(y\right)$$ $$N_n'\left(y\right) = \frac{n}{y} N_n\left(y\right) - N_{n+1}\left(y\right)$$

- $$J_\alpha\left(y\right) N_\alpha'\left(y\right) - J_\alpha'\left(y\right) N_\alpha\left(y\right) = \frac{2}{\pi\, y}$$

- $$T\,\delta_p\left(\vartheta\right) = T\sum_{k=-\infty}^{k=+\infty}\delta\left(\vartheta - kT\right) = \sum_{m=-\infty}^{m=+\infty} e^{j\,m\,2\pi\frac{\vartheta}{T}}$$

INTRODUCTION (28/35)

- $$e^{-j u \hat{\tau}_i \cos(\omega_s t + \psi_i)} = \sum_{m=-\infty}^{m=+\infty} j^{-m} J_m(u \hat{\tau}_i) \, e^{j m (\omega_s t + \psi_i)}$$

- $$\int_0^X J_m^2(a x) \, x \, dx = \frac{X^2}{2} \left[J_m'(a X) \right]^2 + \frac{1}{2} \left[X^2 - \frac{m^2}{a^2} \right] J_m^2(a X)$$

- $$\int_0^X x J_m(a x) J_m(b x) dx = \frac{X}{a^2 - b^2} \left[a J_m(b X) J_{m+1}(a X) - b J_m(a X) J_{m+1}(b X) \right]$$

- $$\mathrm{Erf}[x] = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

- $$\sum_{k=-\infty}^{k=+\infty} \delta\left(u - \frac{2 k \pi}{\Omega_0} \right) = \frac{\Omega_0}{2\pi} \sum_{k=-\infty}^{k=+\infty} e^{j k \Omega_0 u}$$

- $$\sin^2 \theta = \frac{1}{2} - \frac{\cos 2\theta}{2}$$

  $$\sin^3 \theta = \frac{3}{4} \sin\theta - \frac{1}{4} \sin 3\theta$$

$$\sin^4 \theta = \frac{3}{8} - \frac{1}{2} \cos 2\theta + \frac{1}{8} \cos 4\theta$$

$$\sin^5 \theta = \frac{5}{8} \sin\theta - \frac{5}{16} \sin 3\theta + \frac{1}{16} \sin 5\theta$$

INTRODUCTION (29/35)

$$\sin^6\theta = \frac{5}{16} - \frac{15}{32}\cos 2\theta + \frac{3}{16}\cos 4\theta - \frac{1}{32}\cos 6\theta$$

- $$\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$$

$$\sin^7\theta = \frac{35}{64}\sin\theta - \frac{21}{64}\sin 3\theta + \frac{7}{64}\sin 5\theta - \frac{1}{64}\sin 7\theta$$

$$\cos^2\theta = \frac{1}{2} + \frac{\cos 2\theta}{2}$$

$$\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta$$

$$\cos^4\theta = \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$$

$$\cos^5\theta = \frac{5}{8}\cos\theta + \frac{5}{16}\cos 3\theta + \frac{1}{16}\cos 5\theta$$

$$\cos^6\theta = \frac{5}{16} + \frac{15}{32}\cos 2\theta + \frac{3}{16}\cos 4\theta + \frac{1}{32}\cos 6\theta$$

- $$\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$$

$$\cos^7\theta = \frac{35}{64}\cos\theta + \frac{21}{64}\cos 3\theta + \frac{7}{64}\cos 5\theta + \frac{1}{64}\cos 7\theta$$

- $$\int_{\omega=-\infty}^{\omega=+\infty} |\omega| J_m(\omega\hat{\tau}_i) J_m(\omega\hat{\tau}'_i)\, d\omega = \frac{2}{\hat{\tau}_i}\delta(\hat{\tau}_i - \hat{\tau}'_i)$$

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- $$\delta(s - \upsilon t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{j\omega t} \left[\frac{e^{-jks}}{\upsilon} \right]$$

- $$\delta(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{j\omega t} \qquad \delta(\omega) = \int_{-\infty}^{+\infty} e^{-j\omega t} \, dt$$

- $$\frac{K_1'(x_2)}{K_1(x_2)} = \begin{cases} -\dfrac{1}{x_2} & \text{if } |x_2| << 1 \\ -1 & \text{if } |x_2| >> 1 \end{cases}$$

- $$\upsilon \, \delta(\upsilon t) = \delta(t) \qquad \delta(-t) = \delta(t)$$

- $$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t) \, e^{-j\omega t} \, dt$$

- $$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\omega) \, e^{j\omega t} \, dt$$

- $$\delta'(z) = \frac{\delta'(t)}{\upsilon^2} \qquad z = s - \upsilon t$$

- $$\text{FT}\left[\delta'(t) \right] = - j \omega$$

- $$I_1'(x) K_1(x) - I_1(x) K_1'(x) = \frac{1}{x}$$

- $$\int e^{at} \cos(p t) \, dt = \frac{e^{at} \left[a \cos(p t) + p \sin(p t) \right]}{a^2 + p^2} \qquad \int_0^{\infty} e^{-at} \cos(m t) \, dt = \frac{a}{a^2 + m^2} \quad \text{if} \quad a > 0$$

  $$\int e^{at} \sin(p t) \, dt = \frac{e^{at} \left[a \sin(p t) - p \cos(p t) \right]}{a^2 + p^2} \qquad \int_0^{\infty} e^{-at} \sin(m t) \, dt = \frac{m}{a^2 + m^2} \quad \text{if} \quad a > 0$$

- **The trick of the Principal Value (P.V.) of an integral (cf. Chao) is to utilize the property that the divergence on both sides are of opposite signs and, if the integration is taken symmetrically about the singularity so that the divergence on the 2 sides cancel each other, the integral is actually well defined**

$$\text{P.V.} \int_{-\infty}^{+\infty} dx \, \frac{f(x)}{x-a} = \int_{0}^{+\infty} du \, \frac{f(a+u) - f(a-u)}{u}$$

- $$\lim_{t \to +\infty} \frac{\sin(u\,t)}{u} = \pi\,\delta(u)$$

$$\lim_{t \to +\infty} \frac{1-\cos(u\,t)}{u} = \text{P.V.}\left(\frac{1}{u}\right)$$

$$\lim_{t \to +\infty} \frac{\sin^2(u\,t/2)}{u^2} = \frac{\pi\,t}{2}\,\delta(u)$$

- **MKSA units are used, whereas for instance CGS units are used in Chao's book => Conversion from CGS to MKSA**

$$\frac{4\pi}{c} = Z_0 = 120\,\pi\,\Omega$$

$$\frac{e^2}{m_0\,c^2} = r_0 = \text{Classical radius of the particle}$$

- **The engineer convention is also adopted ($e^{j\omega t}$) instead of the physicist's one ($e^{-i\omega t}$)**

(10) Units of physical quantities

Quantity	unit	SI unit	SI derived unit
Capacitance	F (farad)	$m^{-2}\ kg^{-1} s^4 A^2$	C/V
Electric charge	C (coulomb)	As	
Electric potential	V (volt)	$m^2\ kg\ s^{-3} A^{-1}$	W/A
Energy	J (joule)	$m^2\ kg\ s^{-2}$	Nm
Force	N (newton)	$m\ kg\ s^{-2}$	N
Frequency	Hz (hertz)	s^{-1}	
Inductance	H (henry)	$m^2\ kg\ s^{-2} A^{-2}$	Wb/A
Magnetic flux	Wb (weber)	$m^2\ kg\ s^{-2} A^{-1}$	Vs
Magnetic flux density	T (tesla)	$kg\ s^{-2} A^{-1}$	Wb/m^2
Power	W (watt)	$m^2\ kg\ s^{-3}$	J/s
Pressure	Pa (pascal)	$m^{-1}\ kg\ s^{-2}$	N/m^2
Resistance	Ω (ohm)	$m^2\ kg\ s^{-3} A^{-2}$	V/A

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Physical constant	symbol	value	unit
Avogadro's number	N_A	6.0221367×10^{23}	/mol
atomic mass unit ($\frac{1}{12}m(C^{12})$)	m_u or u	$1.6605402 \times 10^{-27}$	kg
Boltzmann's constant	k	1.380658×10^{-23}	J/K
Bohr magneton	$\mu_B = e\hbar/2m_e$	$9.2740154 \times 10^{-24}$	J/T
Bohr radius	$a_0 = 4\pi\epsilon_0\hbar^2/m_e c^2$	$0.529177249 \times 10^{-10}$	m
classical radius of electron	$r_e = e^2/4\pi\epsilon_0 m_e c^2$	$2.81794092 \times 10^{-15}$	m
classical radius of proton	$r_p = e^2/4\pi\epsilon_0 m_p c^2$	$1.5346986 \times 10^{-18}$	m
elementary charge	e	$1.60217733 \times 10^{-19}$	C
fine structure constant	$\alpha = e^2/2\epsilon_0 hc$	1/137.0359895	
$m_u c^2$		931.49432	MeV
mass of electron	m_e	$9.1093897 \times 10^{-31}$	kg
$m_e c^2$		0.51099906	MeV
mass of proton	m_p	$1.6726231 \times 10^{-27}$	kg
$m_p c^2$		938.27231	MeV
mass of neutron	m_n	$1.6749286 \times 10^{-27}$	kg
$m_p c^2$		939.56563	MeV
molar gas constant	$R = N_A k$	8.314510	J/mol K
neutron magnetic moment	μ_n	$-0.96623707 \times 10^{-26}$	J/T
nuclear magneton	$\mu_p = e\hbar/2m_u$	$5.0507866 \times 10^{-27}$	J/T
Planck's constant	h	6.626075×10^{-34}	J s
permeability of vacuum	μ_0	$4\pi \times 10^{-7}$	N/A^2
permittivity of vacuum	ϵ_0	$8.854187817 \times 10^{-12}$	F/m
proton magnetic moment	μ_p	$1.41060761 \times 10^{-26}$	J/T
proton g factor	$g_p = \mu_p/\mu_N$	2.792847386	
speed of light (exact)	c	299792458	m/s
vacuum impedance	$Z_0 = 1/\epsilon_0 c = \mu_0 c$	376.7303	Ω

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