ENVELOPE EQUATIONS

- **Transverse envelope equations**
  - 1-dimensional (8 Slides)
  - 2-dimensional => Coupled oscillations of the transverse beam sizes (8)
- **Longitudinal envelope equation**
  - Longitudinal envelope equation far from transition (4)
  - Evolution of the phase space ellipse near transition, without collective effects (16)
  - Longitudinal envelope equation near transition (6)
  - Bunch rotation with (or without) SC (and or BB impedance) (7)

TRANSVERSE ENVELOPE EQUATIONS (1/16)

- **Consider a particle in an ensemble of particles which obeys the single-particle equations**

$$x' = p_x \qquad p_x' = \frac{F_x(x,s)}{\beta^2 \, E_{total}}$$

- **The total force is** $F_x(x,s) = F_x^{ext} + F_x^{SC}$

- **Let's consider a particle distribution** $f(x, p_x, s)$**. Averaging over the particle distribution, we obtain the equations of motion for the centre of the beam**

$$<x>' = <p_x> \qquad <p_x>' = \frac{<F_x(x,s)>}{\beta^2 \, E_{total}} = \frac{<F_x^{ext}>}{\beta^2 \, E_{total}}$$

as $<F_x^{SC}> = 0$ **, because of Newton's 3rd law**

TRANSVERSE ENVELOPE EQUATIONS (2/16)

- **For a linear machine, one has** $\dfrac{F_x^{ext}}{\beta^2 \, E_{total}} = - K_x (s) \, x$

➔ $< x >'' + K_x (s) < x > = 0$

- **The 2nd moments satisfy the equations**

$$< x^2 >' = 2 < x \, x' > = 2 < x \, p_x >$$

$$< x \, p_x >' = < x' \, p_x > + < x \, p_x' > = < p_x^2 > - K_x (s) < x^2 > + < x \, \frac{F_x^{SC}}{\beta^2 \, E_{total}} >$$

$$< p_x^2 >' = 2 < p_x \, p_x' > = - 2 \, K_x (s) < x \, p_x > + 2 < p_x \, \frac{F_x^{SC}}{\beta^2 \, E_{total}} >$$

TRANSVERSE ENVELOPE EQUATIONS (3/16)

- **To study space-charge effects, we are interested in the position and momentum offsets of the particle from their respective averages, i.e.**

$$\Delta x = x - \langle x \rangle \qquad \Delta p_x = p_x - \langle p_x \rangle$$

➔ $$\langle \Delta x^2 \rangle' = 2 \langle \Delta x \, \Delta p_x \rangle$$

$$\langle \Delta x \, \Delta p_x \rangle' = \langle \Delta p_x^2 \rangle - K_x(s) \langle \Delta x^2 \rangle + \left\langle \Delta x \, \frac{F_x^{SC}}{\beta^2 \, E_{total}} \right\rangle$$

$$\langle \Delta p_x^2 \rangle' = -\, 2 \, K_x(s) \langle \Delta x \, \Delta p_x \rangle + 2 \left\langle \Delta p_x \, \frac{F_x^{SC}}{\beta^2 \, E_{total}} \right\rangle$$

- **Define the rms beam emittance** $\varepsilon_{x,rms} = \sqrt{\langle \Delta x^2 \rangle \langle \Delta p_x^2 \rangle - \langle \Delta x \, \Delta p_x \rangle^2}$

and rms beam size $\sigma_x = \sqrt{\langle \Delta x^2 \rangle}$

TRANSVERSE ENVELOPE EQUATIONS (4/16)

➔ $$<\Delta p_x^2> = \frac{\varepsilon_{x,rms}^2 + <\Delta x \, \Delta p_x>^2}{<\Delta x^2>}$$

$$\sigma'_x = \frac{<\Delta x \, \Delta p_x>}{\sqrt{<\Delta x^2>}}$$

$$\sigma''_x = \frac{<\Delta x \, \Delta p_x>'}{\sqrt{<\Delta x^2>}} - \frac{<\Delta x \, \Delta p_x>^2}{<\Delta x^2>^{3/2}}$$

- **Finally, the transverse envelope equation can be obtained**

$$\sigma''_x + K_x(s)\,\sigma_x - \frac{\varepsilon_{x,rms}^2}{\sigma_x^3} - \frac{<\Delta x \, F_x^{SC}>}{\sigma_x \, \beta^2 \, E_{total}} = 0$$

TRANSVERSE ENVELOPE EQUATIONS (5/16)

- **The SC force was derived in the previous "SC course" (assuming for instance a uniform transverse distribution and a round beam)**

$$\frac{F_x^{SC}}{\beta^2 E_{total}} = \frac{e\,\lambda}{\beta^2 E_{total}\, 2\,\pi\,\varepsilon_0\,\gamma^2}\,\frac{\Delta x}{a^2}$$

$$a = 2\,\sigma_x$$

$$\lambda = \frac{N_b\, e}{l} = N_l\, e$$

=>

$$\frac{F_x^{SC}}{\beta^2 E_{total}} = K_{sc,x2}\,\frac{\Delta x}{a^2}$$

with

$$K_{sc,x2} = \frac{2\,N_l\, r_p}{\beta^2\,\gamma^3}$$

Therefore,

$$\frac{< \Delta x\, F_x^{SC} >}{\beta^2 E_{total}} = \frac{K_{sc,x2}}{4}$$

TRANSVERSE ENVELOPE EQUATIONS (6/16)

- **The 1-dimensional envelope equation can finally be written**

$$a'' + K_x(s)\,a - \frac{\varepsilon_x^2}{a^3} - \frac{K_{sc,x2}}{a} = 0$$

$$a = 2\sigma_x \qquad \varepsilon_x = 4\,\varepsilon_{x,rms}$$

- **Effect of space charge on the equilibrium beam size a_0, in the smooth approximation**

$$K_x = \left(Q_{x0} / R\right)^2$$

TRANSVERSE ENVELOPE EQUATIONS (7/16)

➔ **The equilibrium beam size is therefore found from**

$$\left(\frac{Q_{x0}}{R}\right)^2 a_0 - \frac{K_{sc,x2}}{a_0} - \frac{\varepsilon_x^2}{a_0^3} = 0$$

which yields $$a_0^2 = \frac{\varepsilon_x R}{Q_{x0}} \left(\kappa + \sqrt{1+\kappa^2} \right)$$ $$\kappa = \frac{K_{sc,x2} R}{2 \varepsilon_x Q_{x0}}$$

- **The beam size is significantly perturbed by the space-charge force when** $\kappa \geq 1$

- **If the beam size becomes larger than the vacuum chamber aperture, there will be a beam loss**

TRANSVERSE ENVELOPE EQUATIONS (8/16)

- **For weak beam intensities, i.e.** $\kappa << 1$

$$a_0^2 = a_{00}^2 + \Delta a_{00}^2$$

with $a_{00}^2 = \dfrac{\varepsilon_x R}{Q_{x0}}$ $\quad \Delta a_{00}^2 = \kappa\, a_{00}^2$

- **The parameter** a_{00} **describes the beam size in the absence of space charge. Interpreting** Δa_{00} **as a perturbation on the single-particle tune according to** $a_0^2 = \varepsilon_x R / (Q_{x0} + \Delta Q_x)$**, gives an expression for the shift of the single-particle tune due to space charge**

$$\Delta Q_x = -\frac{K_{sc,x2} R}{2 \varepsilon_x}$$

➔ **It is ½ of the result found in the “SC course” with** $a = \sqrt{2}\, \sigma_x$ **(as expected)**

TRANSVERSE ENVELOPE EQUATIONS (9/16)

- **Let's come back to the general case, i.e. consider a beam with unequal transverse beam sizes => The envelope equations are therefore given by (we saw in the "SC course", that in the SC force, a^2 must be replaced by $a\left(a+b\right)/2$)**

$$a'' + K_x\, a - \frac{2K_{sc,x2}}{a+b} - \frac{\varepsilon_x^2}{a^3} = 0$$

$$b'' + K_y\, b - \frac{2K_{sc,x2}}{a+b} - \frac{\varepsilon_y^2}{b^3} = 0$$

$$a = 2\sigma_x \qquad \varepsilon_x = 4\,\varepsilon_{x,rms}$$

$$b = 2\sigma_y \qquad \varepsilon_y = 4\,\varepsilon_{y,rms}$$

TRANSVERSE ENVELOPE EQUATIONS (10/16)

➔ **Both transverse planes have thus to be treated jointly for high-intensity beams due to space-charge coupling**

- The beam may execute some collective motion on top of equilibrium beam sizes a_0 and b_0
- Let the horizontal and vertical beam sizes be

$$a(s) = a_0 - \Delta a(s) \,,$$

$$b(s) = b_0 + \Delta b(s)$$

where the perturbations Δa and Δb are considered small with respect to the equilibrium sizes

- Linearizing yields

$$\Delta a'' + K_a \, \Delta a = K \, \Delta b$$

$$\Delta b'' + K_b \, \Delta b = K \, \Delta a$$

$$K = \frac{2 K_{sc,x2}}{\left(a_0 + b_0 \right)^2}$$

$$K_a = 4 K_x - \frac{2 K_{sc,x2} \left(2 a_0 + 3 b_0 \right)}{a_0 \left(a_0 + b_0 \right)^2}$$

$$K_b = 4 K_y - \frac{2 K_{sc,x2} \left(2 b_0 + 3 a_0 \right)}{b_0 \left(a_0 + b_0 \right)^2}$$

➔ **The transverse beam sizes execute coupled oscillations**

TRANSVERSE ENVELOPE EQUATIONS (11/16)

- **The equilibrium beam sizes a_0 and b_0 are found from the following equations**

$$K_x\, a_0 - \frac{2K_{sc,x2}}{a_0 + b_0} - \frac{\varepsilon_x^2}{a_0^3} = 0$$

$$K_y\, b_0 - \frac{2K_{sc,x2}}{a_0 + b_0} - \frac{\varepsilon_y^2}{b_0^3} = 0$$

- **Using the smooth approximation**

$$K_x = \left(Q_{x0} / R \right)^2 \qquad K_a = \left(Q_a / R \right)^2$$

$$K_y = \left(Q_{y0} / R \right)^2 \qquad K_b = \left(Q_b / R \right)^2$$

and assuming small tune shifts, yields

TRANSVERSE ENVELOPE EQUATIONS (12/16)

$$Q_a = 2Q_{x0} + \Delta Q_a = 2Q_{x0} - \frac{K_{sc,x2} R^2 \left(2a_0 + 3b_0 \right)}{2Q_{x0} a_0 \left(a_0 + b_0 \right)^2}$$

$$Q_b = 2Q_{y0} + \Delta Q_b = 2Q_{y0} - \frac{K_{sc,x2} R^2 \left(2b_0 + 3a_0 \right)}{2Q_{y0} b_0 \left(a_0 + b_0 \right)^2}$$

- **The coupled equations can be re-written**

$$\frac{d^2 \Delta a}{d\phi^2} + Q_a^2 \Delta a = K R^2 \Delta b$$

$$\phi = \Omega_0 t$$

$$\frac{d^2 \Delta b}{d\phi^2} + Q_b^2 \Delta b = K R^2 \Delta a$$

TRANSVERSE ENVELOPE EQUATIONS (13/16)

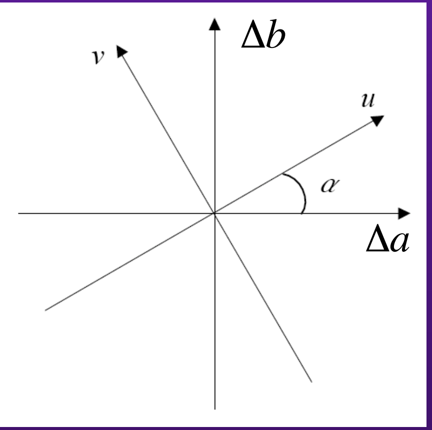
- **Far from the coupling resonance $Q_a = Q_b$, the solutions of the homogeneous equations (of the coupled oscillations) are given by**

$$\Delta a = \Delta a_0 \, e^{j Q_a \phi} \qquad \Delta b = \Delta b_0 \, e^{j Q_b \phi}$$

- **In the presence of coupling, the coupled oscillations can be solved by searching the normal (i.e. decoupled) modes (*u*,*v*) linked by a simple rotation**

$$\begin{pmatrix} \Delta a \\ \Delta b \end{pmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

- **The equations of the 2 normal modes can be found**



$$\frac{d^2 u}{d\phi^2} + Q_u^2 \, u = 0 \qquad \frac{d^2 v}{d\phi^2} + Q_v^2 \, v = 0$$

TRANSVERSE ENVELOPE EQUATIONS (14/16)

with (assuming small tune shifts)

$$Q_u = Q_a - \frac{|C|}{2} \tan\alpha \qquad Q_v = Q_b + \frac{|C|}{2} \tan\alpha$$

$$\tan(2\alpha) = \frac{|C|}{\Delta} \qquad |C| = \frac{R^2 K}{Q_0} \qquad \Delta = Q_b - Q_a \qquad Q_{x0} \approx Q_{y0} \approx Q_0$$

- **The solutions for the decoupled modes are** $u = U\, e^{j Q_u \phi}$ $v = V\, e^{j Q_v \phi}$

where (*U*,*V*) are constants of motion, which depend on the initial conditions ➔

$$\Delta a = \Delta a_0\, e^{j Q_u \phi} \cos\alpha - \Delta b_0\, e^{j Q_v \phi} \sin\alpha$$

$$\Delta b = \Delta a_0\, e^{j Q_u \phi} \sin\alpha + \Delta b_0\, e^{j Q_v \phi} \cos\alpha$$

TRANSVERSE ENVELOPE EQUATIONS (15/16)

$$\left|\Delta a\right|^2 = \Delta a \, \Delta a^*$$

- **Therefore,**

$$\left|\Delta a\right|^2 = \left|\Delta a_0\right|^2 \cos^2\alpha + \left|\Delta b_0\right|^2 \sin^2\alpha - \left|\Delta a_0 \, \Delta b_0\right| \sin(2\alpha) \cos\left[(Q_v - Q_u)\,\phi\right]$$

$$\left|\Delta b\right|^2 = \left|\Delta a_0\right|^2 \sin^2\alpha + \left|\Delta b_0\right|^2 \cos^2\alpha + \left|\Delta a_0 \, \Delta b_0\right| \sin(2\alpha) \cos\left[(Q_v - Q_u)\,\phi\right]$$

➔ $$\left|\Delta a\right|^2 + \left|\Delta b\right|^2 = \left|\Delta a_0\right|^2 + \left|\Delta b_0\right|^2$$

- **Furthermore, using the fact that**

$$\cos(2\alpha) = \cos\left[\arctan\left(\frac{|C|}{\Delta}\right)\right] = \left(1 + \frac{|C|^2}{\Delta^2}\right)^{-1/2}$$

➔ $$\sin^2\alpha = \frac{|C|^2 / 2}{\Delta^2 + |C|^2 + \Delta\sqrt{\Delta^2 + |C|^2}}$$

TRANSVERSE ENVELOPE EQUATIONS (16/16)

and, averaging over time (i.e. Φ), it yields (when the resonance is crossed)

$$\left|\Delta a\right|^2 = \left|\Delta a_0\right|^2 - \left(\left|\Delta a_0\right|^2 - \left|\Delta b_0\right|^2\right) \frac{\left|C\right|^2 / 2}{\Delta^2 + \left|C\right|^2 + \Delta \sqrt{\Delta^2 + \left|C\right|^2}}$$

$$\left|\Delta b\right|^2 = \left|\Delta b_0\right|^2 + \left(\left|\Delta a_0\right|^2 - \left|\Delta b_0\right|^2\right) \frac{\left|C\right|^2 / 2}{\Delta^2 + \left|C\right|^2 + \Delta \sqrt{\Delta^2 + \left|C\right|^2}}$$

LONGITUDINAL ENVELOPE EQUATION (1/33)

- **Consider as canonical coordinates** $\Delta\phi = \phi - \phi_s$ $\quad p = \dfrac{h\,\Omega_0\,\eta}{\beta^2\,\gamma\,E_{rest}}\,\Delta E$

$$\Delta\phi = \omega_{RF}\,\Delta t = h\,\Omega_0\,\Delta t \qquad \Delta E = \beta^2\,E_{total}\,\delta$$

- **The single-particle equations to be solved (far away from transition) are**

$$\Delta\dot{\phi} = p \qquad \dot{p} = F_l$$

- **The total force is** $F_l = F_l^{ext} + F_l^{SC}$

- **Without SC one has** $F_l^{ext} = -\,\omega_{s0}^2\,\Delta\phi$ ➔ $\Delta\ddot{\phi} + \omega_{s0}^2\,\Delta\phi = 0$

LONGITUDINAL ENVELOPE EQUATION (2/33)

- **With SC, one can follow exactly what was done in the transverse plane, making the following replacements**

$$\Delta x \rightarrow \Delta\phi$$

$$p_x \rightarrow p$$

$$K_x \rightarrow \omega_{s0}^2$$

$$\sigma_x = \sqrt{<\Delta x^2>} \rightarrow \tilde{\phi} = \sqrt{<\Delta\phi^2>}$$

$$\varepsilon_x = \sqrt{<\Delta x^2><\Delta p_x^2> - <\Delta x\ \Delta p_x>^2} \rightarrow E_0 = \sqrt{<\Delta\phi^2><p^2> - <\Delta\phi\ p>^2}$$

- **Finally, the longitudinal envelope equation can be obtained**

$$\ddot{\tilde{\phi}} + \omega_{s0}^2\ \tilde{\phi} - \frac{E_0^2}{\tilde{\phi}^3} - \frac{<\Delta\phi\ F_l^{SC}>}{\tilde{\phi}} = 0$$

LONGITUDINAL ENVELOPE EQUATION (3/33)

- Using the fact that $F_l^{SC} = -\,\eta_{SC}\,\omega_{s0}^2\,\Delta\phi$, with $\eta_{SC} = \dfrac{K_{sc,l}}{\Delta\hat{\phi}^3}$ (see "SC course")

$$K_{sc,l} = \frac{3\pi N_b\, r_p\, E_{rest}\, g_0\, h^2\, Sgn(\eta)}{R\,\gamma^2\, e\,\hat{V}_{RF}\left|\cos\phi_s\right|}$$

$$g_0 = 1 + 2\ln\left(\frac{b}{a}\right)$$

$$\Delta\hat{\phi} = \pi\, f_0\, h\, \tau_b$$

For a parabolic distribution

Half bunch length

and the fact that $\dfrac{<\Delta\phi^2>}{\tilde{\phi}} = \tilde{\phi}$

$$\rightarrow \quad \ddot{\tilde{\phi}} + \omega_{s0}^2\,\tilde{\phi} - \frac{E_0^2}{\tilde{\phi}^3} + \frac{\omega_{s0}^2\,K_{sc,l}}{\Delta\hat{\phi}^3} = 0$$

Let's now express E_0 as a function of the usual longitudinal emittance

$\varepsilon_{l,rms}\,[\mathrm{eVs}]$

LONGITUDINAL ENVELOPE EQUATION (4/33)

$$\Delta\phi = \omega_{RF}\ t = h\ \Omega_0\ t$$

$$p = \frac{h\ \Omega_0\ \eta}{\beta^2\ \gamma\ E_{rest}}\ \Delta E$$

➔ $$p\ \Delta\phi = \frac{h^2\ \Omega_0^2\ \eta}{\beta^2\ \gamma\ E_{rest}}\ \Delta E\ t$$

➔ $$E_0 = \frac{h^2\ \Omega_0^2\ \left|\eta\right|}{\beta^2\ \gamma\ E_{rest}} \times \frac{\varepsilon_{l,rms}\left[\mathrm{eVs}\right]}{\pi}$$

- **Now, let's convert the envelope equation to one for the half bunch length $\Delta\hat{\phi}$. For a parabolic distribution**

$$\Delta\hat{\phi} = \sqrt{5}\ \tilde{\phi}$$

LONGITUDINAL ENVELOPE EQUATION (5/33)

➔ $$\Delta\ddot{\hat{\phi}} + \omega_{s0}^2 \, \Delta\hat{\phi} + \frac{\omega_{s0}^2 \, K_{sc,l}}{\Delta\hat{\phi}^2} - \frac{25 \, E_0^2}{\Delta\hat{\phi}^3} = 0$$

Envelope equation for the half bunch length far from transition

- **Let's now derive the evolution of the phase space ellipse near transition (before deriving the envelope equation near transition)**

Synchrotron oscillations equation

$$\frac{d\,\Delta\phi}{d\,t} = \frac{h\,\eta\,\Omega_0}{\beta^2 \, E_{total}} \, \Delta E$$

$$\frac{d\,\Delta E}{d\,t} = \frac{e\,\hat{V}_{RF}\,\Omega_0}{2\pi} \left[\sin\left(\phi_s + \Delta\phi\right) - \sin\phi_s \right] \approx \frac{e\,\hat{V}_{RF}\,\Omega_0}{2\pi} \cos\phi_s \, \Delta\phi$$

LONGITUDINAL ENVELOPE EQUATION (6/33)

$$\rightarrow \quad \frac{d}{dt}\left(\frac{\beta^2\, E_{total}}{h\,\eta\,\Omega_0}\,\frac{d\,\Delta\phi}{dt}\right) - \frac{e\,\hat{V}_{RF}\,\Omega_0}{2\pi}\cos\phi_s\,\Delta\phi = 0$$

with β , E_{total} , η , Ω_0 which depend on time t

Approximation: We neglect the slow time variations of all the parameters except $\frac{\eta}{E_{total}}$

$$\rightarrow \quad \frac{d}{dt}\left(\frac{E_{total}}{\eta}\,\frac{d\,\Delta\phi}{dt}\right) - \frac{h\,e\,\hat{V}_{RF}\,\Omega_0^2\cos\phi_s}{2\pi\,\beta^2}\,\Delta\phi = 0$$

Assumed to be time independent

LONGITUDINAL ENVELOPE EQUATION (7/33)

Furthermore, $\gamma = \gamma_t + \dot{\gamma}\, t$ **, with t < 0 below transition and t > 0 above transition**

➔ $$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2} \approx \frac{2\,\dot{\gamma}\,t}{\gamma_t^3}$$ **and** $$E_{total} = \gamma\, E_{rest} \approx \gamma_t\, E_{rest}$$

➔ $$\frac{\eta}{E_{total}} \approx \frac{2\,\dot{\gamma}\,t}{\gamma_t^4\, E_{rest}}$$

➔ $$\frac{d}{d\,t}\left(\frac{1}{\omega_{s0}^2}\,\frac{d\,\Delta\phi}{d\,t}\right) + \Delta\phi = 0$$ **with** $$\omega_{s0} = \Omega_0 \left(-\frac{e\,\hat{V}_{\mathrm{RF}}\,h\,\eta\,\cos\phi_s}{2\,\pi\,\beta^2\,E_{total}}\right)^{1/2}$$

But here, this should be considered as a definition only, as the beam particle do not make synchrotron oscillations and therefore it loses its meaning of oscillation frequency

LONGITUDINAL ENVELOPE EQUATION (8/33)

One can write $\omega_{s0}^2 = \frac{|t|}{T_c^3}$, with $T_c = \left(\frac{\beta^2 E_{rest} \gamma_t^4}{4 \pi f_0^2 \dot{\gamma} h e \hat{V}_{RF} |\cos \phi_s|} \right)^{1/3}$

It is called the nonadiabatic time

Physical meaning of the nonadiabatic time: When the time is close enough to transition, the particle will not be able to catch up with the rapid changing of the bucket shape

➔ One has to solve

$$\frac{d}{dt}\left(\frac{T_c^3}{|t|} \frac{d\,\Delta\phi}{dt} \right) + \Delta\phi = 0$$

LONGITUDINAL ENVELOPE EQUATION (9/33)

Defining a new time variable, and considering only t > 0 for the moment

$$y = \int_0^x \sqrt{u}\; du = \frac{2}{3}\, x^{3/2}$$

$$x = \frac{t}{T_c}$$

➔ One has to solve

$$\frac{d}{dx}\left(\frac{1}{x}\, \frac{d\,\Delta\phi}{dx} \right) + \Delta\phi = 0$$

Let's call $\Delta\phi = \varphi\, y^{2/3}$

➔

$$\frac{d^2\varphi}{dy^2} + \frac{1}{y}\,\frac{d\varphi}{dy} + \left[1 - \frac{\left(\frac{2}{3}\right)^2}{y^2} \right] \varphi = 0$$

LONGITUDINAL ENVELOPE EQUATION (10/33)

The solution of this equation can be written

$$\varphi = C_1\, J_{2/3}\left(y \right) + C_2\, N_{2/3}\left(y \right) = C_3 \left[J_{2/3}\left(y \right) \cos\chi + N_{2/3}\left(y \right) \sin\chi \right]$$

Constants to be determined from the initial conditions

with $J_{2/3}$ the Bessel function of the 1st kind (or simply the Bessel function), and $N_{2/3}$ the Bessel function of the 2nd kind (also called Weber or Neumann function)

➔ $$\Delta\phi = b\, x \left[J_{2/3}\left(y \right) \cos\chi + N_{2/3}\left(y \right) \sin\chi \right]$$

Constants to be determined from the initial conditions

LONGITUDINAL ENVELOPE EQUATION (11/33)

$$\Delta E = \beta^2 \; E_{total} \; \delta$$

$$\delta = \frac{\Delta p}{p_0}$$

and

$$\Delta E = \frac{\beta^2 \, E_{total}}{h \, \Omega_0 \, \eta} \frac{d \, \Delta\phi}{d \, t}$$

➔ $$\delta = \frac{1}{h \, \Omega_0 \, \eta} \frac{d \, \Delta\phi}{d \, t} = \frac{\gamma_t^3}{2 \, \dot{\gamma} \, t \, h \, \Omega_0} \frac{d \, \Delta\phi}{d \, t}$$

➔ $$\delta = \frac{\gamma_t^3}{2 \, \dot{\gamma} \, t \, h \, \Omega_0} \left\{ \frac{\Delta\phi}{x \, T_c} + \frac{b \, x^{3/2}}{T_c} \left[\left(\frac{2 \, J_{2/3}}{3 \, y} - J_{5/3} \right) \cos\chi + \left(\frac{2 \, N_{2/3}}{3 \, y} - N_{5/3} \right) \sin\chi \right] \right\}$$

using the fact that

$$J'_n(y) = \frac{n}{y} J_n(y) - J_{n+1}(y)$$

$$N'_n(y) = \frac{n}{y} N_n(y) - N_{n+1}(y)$$

LONGITUDINAL ENVELOPE EQUATION (12/33)

Now the idea is, with the 2 equations

$$\Delta\phi = b\, x \left[J_{2/3}(y) \cos\chi + N_{2/3}(y) \sin\chi \right]$$

$$\delta = \frac{\gamma_t^3}{2\,\dot{\gamma}\, t\, h\, \Omega_0} \left\{ \frac{\Delta\phi}{x\, T_c} + \frac{b\, x^{3/2}}{T_c} \left[\left(\frac{2\, J_{2/3}}{3\, y} - J_{5/3} \right) \cos\chi + \left(\frac{2\, N_{2/3}}{3\, y} - N_{5/3} \right) \sin\chi \right] \right\}$$

to solve for $\cos\chi$ **and** $\sin\chi$**, and then write the equation** $\cos^2\chi + \sin^2\chi = 1$ **to derive an equation linking** $\Delta\phi$ **and** δ

➔ $$\alpha_{\phi\phi}\, \Delta\phi^2 + 2\, \alpha_{\phi\delta}\, \Delta\phi\, \delta + \alpha_{\delta\delta}\, \delta^2 = 1$$

LONGITUDINAL ENVELOPE EQUATION (13/33)

with

$$\alpha_{\phi\phi} = \frac{1}{\det^2 b^2 x^2} \left\{ \begin{array}{l} \dfrac{J_{2/3}^2}{x^3} + \left(\dfrac{2\, J_{2/3}}{3\, y} - J_{5/3} \right)^2 + \dfrac{2 J_{2/3}}{x^{3/2}} \left(\dfrac{2\, J_{2/3}}{3\, y} - J_{5/3} \right) \\ + \left(\dfrac{2\, N_{2/3}}{3\, y} - N_{5/3} \right)^2 + \dfrac{N_{2/3}^2}{x^3} + \dfrac{2 N_{2/3}}{x^{3/2}} \left(\dfrac{2\, N_{2/3}}{3\, y} - N_{5/3} \right) \end{array} \right\}$$

$$\alpha_{\delta\delta} = \frac{1}{\det^2 b^2 x^3} \left(\frac{2\, \dot{\gamma}\, t\, h\, \Omega_0\, T_c}{\gamma_t^3} \right)^2 \left(J_{2/3}^2 + N_{2/3}^2 \right)$$

$$\alpha_{\phi\delta} = \frac{1}{\det^2} \left\{ \begin{array}{l} - \dfrac{2\, J_{2/3}^2\, T_c\, h\, \Omega_0\, \dot{\gamma}\, t}{b^2\, x^4\, \gamma_t^3} - \dfrac{J_{2/3}\, T_c}{b^2\, x^{5/2}\, \gamma_t^3} \left(\dfrac{2\, J_{2/3}}{3\, y} - J_{5/3} \right) 2\, \dot{\gamma}\, t\, h\, \Omega_0 \\ - \dfrac{2\, N_{2/3}^2\, T_c\, h\, \Omega_0\, \dot{\gamma}\, t}{b^2\, x^4\, \gamma_t^3} - \dfrac{N_{2/3}\, T_c}{b^2\, x^{5/2}\, \gamma_t^3} \left(\dfrac{2\, N_{2/3}}{3\, y} - N_{5/3} \right) 2\, \dot{\gamma}\, t\, h\, \Omega_0 \end{array} \right\}$$

$$\det = J_{2/3} \left(\frac{2\, N_{2/3}}{3\, y} - N_{5/3} \right) - N_{2/3} \left(\frac{2\, J_{2/3}}{3\, y} - J_{5/3} \right)$$

LONGITUDINAL ENVELOPE EQUATION (14/33)

Using the fact that

$$J_\alpha(y)\,N'_\alpha(y) - J'_\alpha(y)\,N_\alpha(y) = \frac{2}{\pi\, y}$$

➔

$$\det = J_{2/3}\left(\frac{2\,N_{2/3}}{3\,y} - N_{5/3}\right) - N_{2/3}\left(\frac{2\,J_{2/3}}{3\,y} - J_{5/3}\right) = J_{2/3}\,N'_{2/3} - N_{2/3}\,J'_{2/3} = \frac{2}{\pi\, y}$$

➔

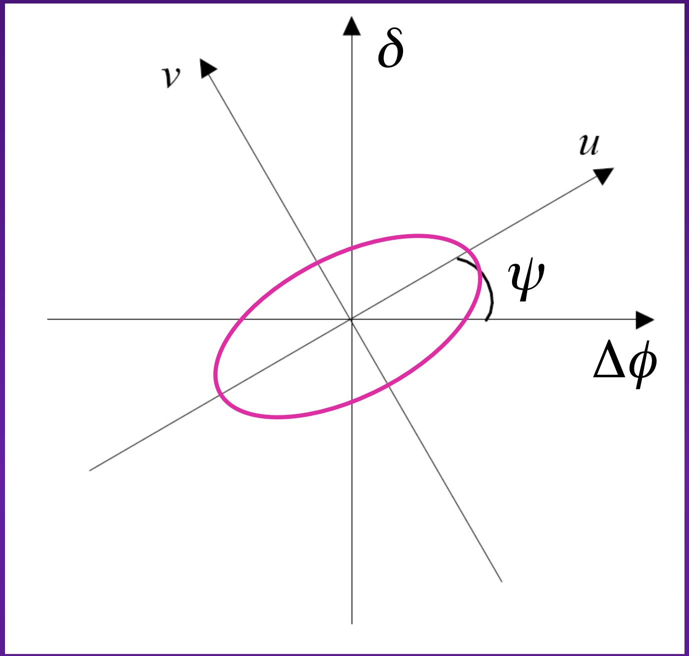
$$\alpha_{\phi\phi} = \frac{\pi^2}{9\,b^2\,x^2}\left\{\left(2\,J_{2/3} - \frac{3\,y}{2}\,J_{5/3}\right)^2 + \left(2\,N_{2/3} - \frac{3\,y}{2}\,N_{5/3}\right)^2\right\}$$

$$\alpha_{\delta\delta} = \frac{\pi^2\,x^2}{9\,b^2}\left(\frac{2\,\dot{\gamma}\,h\,\Omega_0\,T_c^2}{\gamma_t^3}\right)^2\left(J_{2/3}^2 + N_{2/3}^2\right)$$

$$\alpha_{\phi\delta} = \frac{\pi^2}{9\,b^2}\left(\frac{2\,\dot{\gamma}\,h\,\Omega_0\,T_c^2}{\gamma_t^3}\right)\left[N_{2/3}\left(\frac{3\,y}{2}\,N_{5/3} - 2\,N_{2/3}\right) - J_{2/3}\left(2\,J_{2/3} - \frac{3\,y}{2}\,J_{5/3}\right)\right]$$

LONGITUDINAL ENVELOPE EQUATION (15/33)

Let's now compute the emittance of the tilted ellipse



Method: Let's find (u , v) where the ellipse is upright

$$\begin{pmatrix} \Delta\phi \\ \delta \end{pmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

LONGITUDINAL ENVELOPE EQUATION (16/33)

=> The ellipse is upright if

$$\psi = \frac{1}{2} \arctan\left(\frac{2\,\alpha_{\phi\delta}}{\alpha_{\phi\phi} - \alpha_{\delta\delta}} \right)$$

It is the tilt angle of the ellipse

Using the fact that

$$\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}} \qquad \sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$$

the equation of the ellipse can be written

$$\frac{1}{2}\left[\alpha_{\phi\phi} + \alpha_{\delta\delta} + \sqrt{\left(\alpha_{\phi\phi} - \alpha_{\delta\delta}\right)^2 + 4\,\alpha_{\phi\delta}^2}\right] u^2 + \frac{1}{2}\left[\alpha_{\phi\phi} + \alpha_{\delta\delta} - \sqrt{\left(\alpha_{\phi\phi} - \alpha_{\delta\delta}\right)^2 + 4\,\alpha_{\phi\delta}^2}\right] v^2 = 1$$

LONGITUDINAL ENVELOPE EQUATION (17/33)

The area of the ellipse is thus

$$A = \pi \sqrt{\frac{2}{\alpha_{\phi\phi} + \alpha_{\delta\delta} + \sqrt{\left(\alpha_{\phi\phi} - \alpha_{\delta\delta}\right)^2 + 4\,\alpha_{\phi\delta}^2}}} \sqrt{\frac{2}{\alpha_{\phi\phi} + \alpha_{\delta\delta} - \sqrt{\left(\alpha_{\phi\phi} - \alpha_{\delta\delta}\right)^2 + 4\,\alpha_{\phi\delta}^2}}}$$

$$\rightarrow \quad A = \frac{\pi}{\sqrt{\alpha_{\phi\phi}\,\alpha_{\delta\delta} - \alpha_{\phi\delta}^2}}$$

and

$$A = \varepsilon_l\left[\mathrm{eV}s\right]\frac{h\,\Omega_0}{\beta^2\,E_{total}}$$

Using the fact that

$$J_{2/3}\left(\frac{3\,y\,N_{5/3}}{2} - 2\,N_{2/3}\right) + N_{2/3}\left(2\,J_{2/3} - \frac{3\,y\,J_{5/3}}{2}\right) = -\frac{3}{\pi}$$

LONGITUDINAL ENVELOPE EQUATION (18/33)

=> The remaining unknown ***b*** can be found and is given by

$$b = \sqrt{\frac{2\,\varepsilon_l\,h^2\,\Omega_0^2\,\dot{\gamma}\,T_c^2}{3\,m_0\,c^2\,\beta^2\,\gamma_t^4}}$$

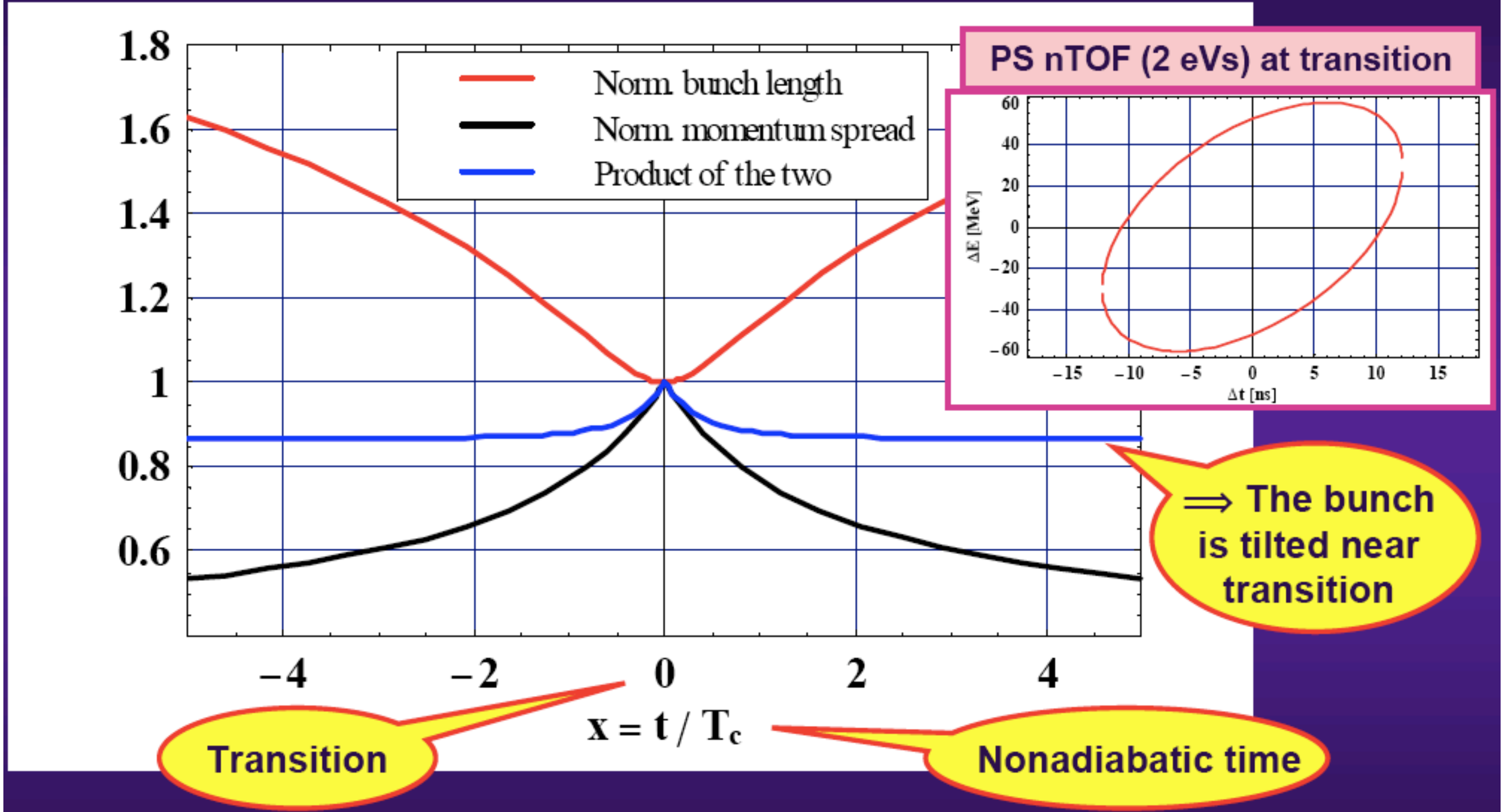
One now has all the parameters of the ellipse which can be plotted at any time ***t*** => The evolution of the bunch length, the energy spread and the phase space ellipse can be studied near transition

$$\Delta\phi_{\max} = \frac{\sqrt{\alpha_{\delta\delta}}}{\sqrt{\alpha_{\phi\phi}\,\alpha_{\delta\delta} - \alpha_{\phi\delta}^2}}$$

$$\delta_{\max} = \frac{\sqrt{\alpha_{\phi\phi}}}{\sqrt{\alpha_{\phi\phi}\,\alpha_{\delta\delta} - \alpha_{\phi\delta}^2}}$$

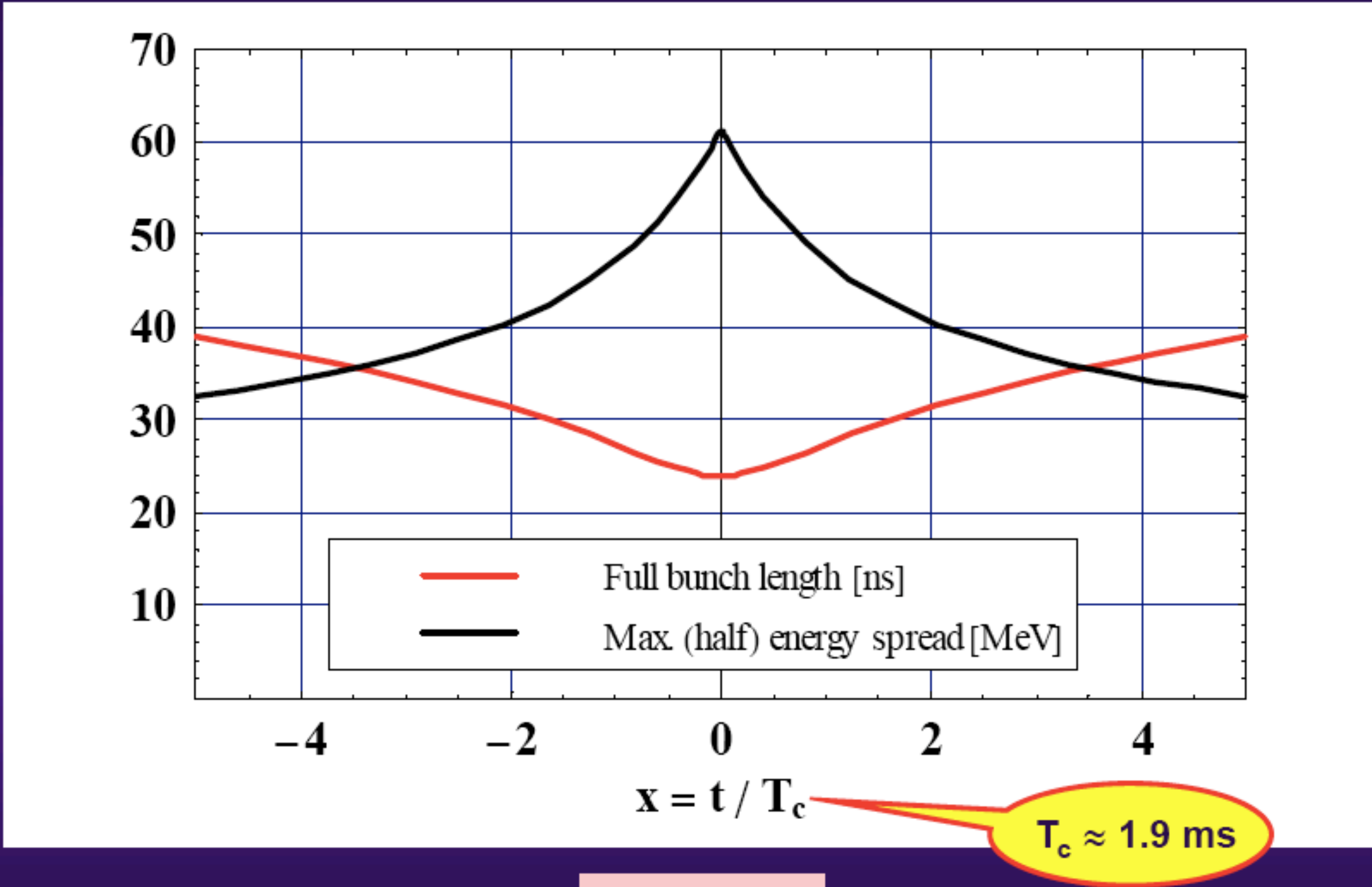
Properties in an ellipse (see "Introduction")

LONGITUDINAL ENVELOPE EQUATION (19/33)



LONGITUDINAL ENVELOPE EQUATION (20/33)

Case of the CERN PS nTOF bunch without SC



$\varepsilon_L = 2$ eVs

LONGITUDINAL ENVELOPE EQUATION (21/33)

- Let's now concentrate on the derivation of the envelope equation **NEAR TRANSITION.** Far from transition, the envelope equation was found to be

$$\Delta\ddot{\hat{\phi}} + \omega_{s0}^2 \, \Delta\hat{\phi} + \frac{\omega_{s0}^2 \, K_{sc,l}}{\Delta\hat{\phi}^2} - \frac{25 \, E_0^2}{\Delta\hat{\phi}^3} = 0$$

which can also be written

$$\frac{1}{\omega_{s0}^2} \frac{d^2 \tau_b}{d t^2} + \tau_b + \frac{K_{sc,l1}}{\tau_b^2} - \frac{S_1}{\tau_b^3} = 0$$

with

$$K_{sc,l1} = \frac{K_{sc,l}}{\pi^3 \, h^3 \, f_0^3} = \frac{3 \, N_b \, r_p \, E_{rest} \, g_0 \, Sgn(\eta)}{\pi^2 \, h \, f_0^3 \, R \, \gamma_t^2 \, e \, \hat{V}_{RF} \left| \cos\phi_s \right|}$$

$$S_1 = \frac{16 \, \eta^2 \, \varepsilon_l^2}{\pi^2 \, \beta^4 \, \gamma^2 \, E_{rest}^2 \, \omega_{s0}^2}$$

$$\varepsilon_l = 5 \, \varepsilon_{l,rms}$$

LONGITUDINAL ENVELOPE EQUATION (22/33)

Close to transition

$$x = \frac{t}{T_c}$$

$$\omega_{s0}^2 = \frac{|t|}{T_c^3}$$

➔

$$\omega_{s0}^2 = \frac{|x|}{T_c^2}$$

➔

$$\frac{1}{|x|}\frac{d^2 \tau_b}{d x^2} + \tau_b + \frac{K_{sc,l1}}{\tau_b^2} - \frac{S_1}{\tau_b^3} = 0$$

Knowing that, close to transition, one has to make the substitution (as seen in the previous slides)

$$\frac{1}{|x|}\frac{d^2 \tau_b}{d x^2} \quad ➔ \quad \frac{d}{d x}\left(\frac{1}{|x|}\frac{d \tau_b}{d x}\right)$$

LONGITUDINAL ENVELOPE EQUATION (23/33)

➔ One has to solve

$$\frac{d}{dx}\left(\frac{1}{|x|}\frac{d\tau_b}{dx}\right)+\tau_b+\frac{K_{sc,l1}}{\tau_b^2}-\frac{S_1}{\tau_b^3}=0$$

which can also be written

$$\frac{d}{dx}\left(\frac{1}{|x|}\frac{d\tau_{bns}}{dx}\right)+\tau_{bns}+\frac{10^{27}K_{sc,l1}}{\tau_{bns}^2}-\frac{10^{36}|x|S_2}{\tau_{bns}^3}=0$$

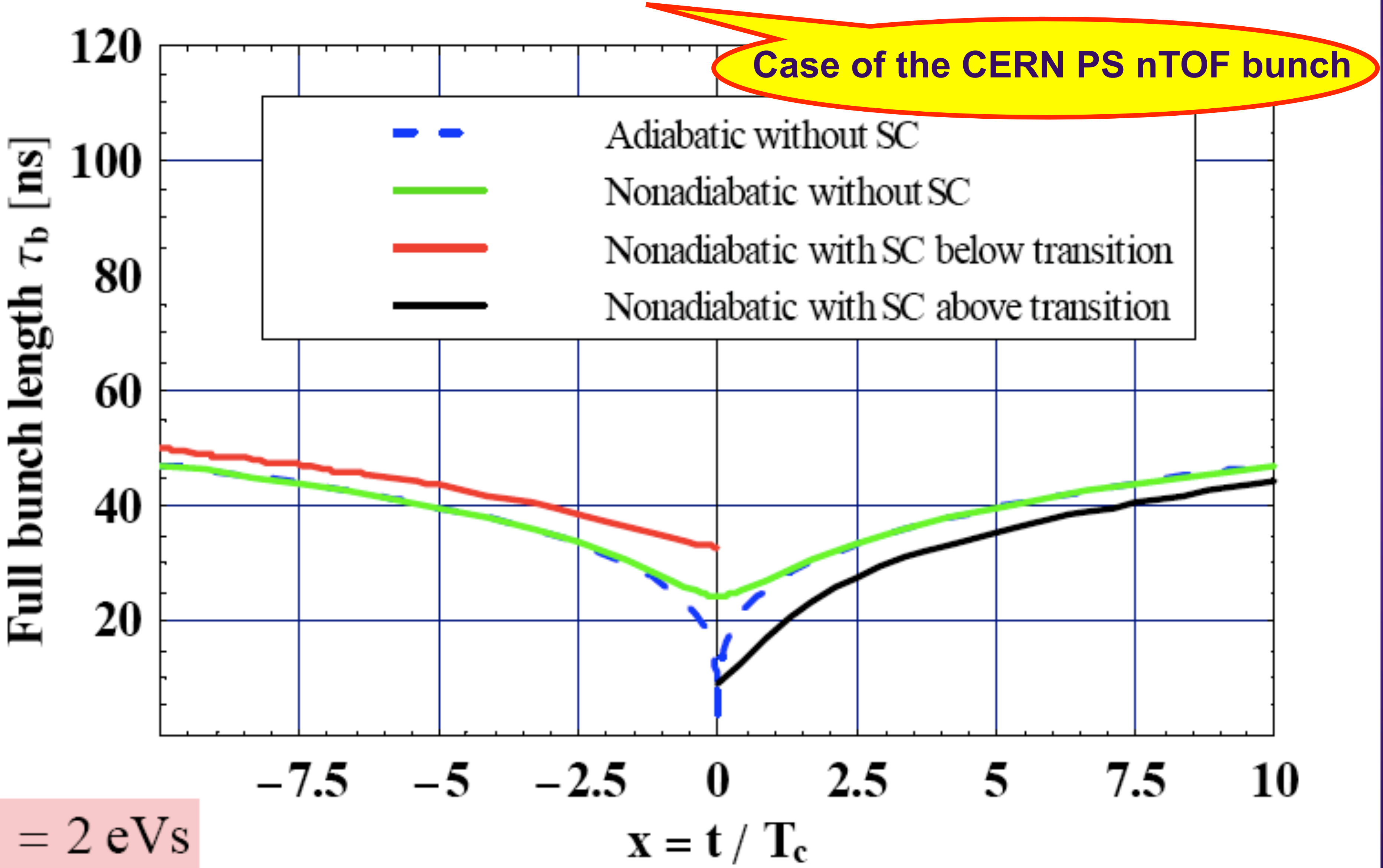
with

$$S_2=\frac{64\,\varepsilon_l^2\,\dot{\gamma}^2\,T_c^4}{\pi^2\,\beta^4\,\gamma_t^8\,E_{rest}^2}$$

Full (4σ) bunch length in ns

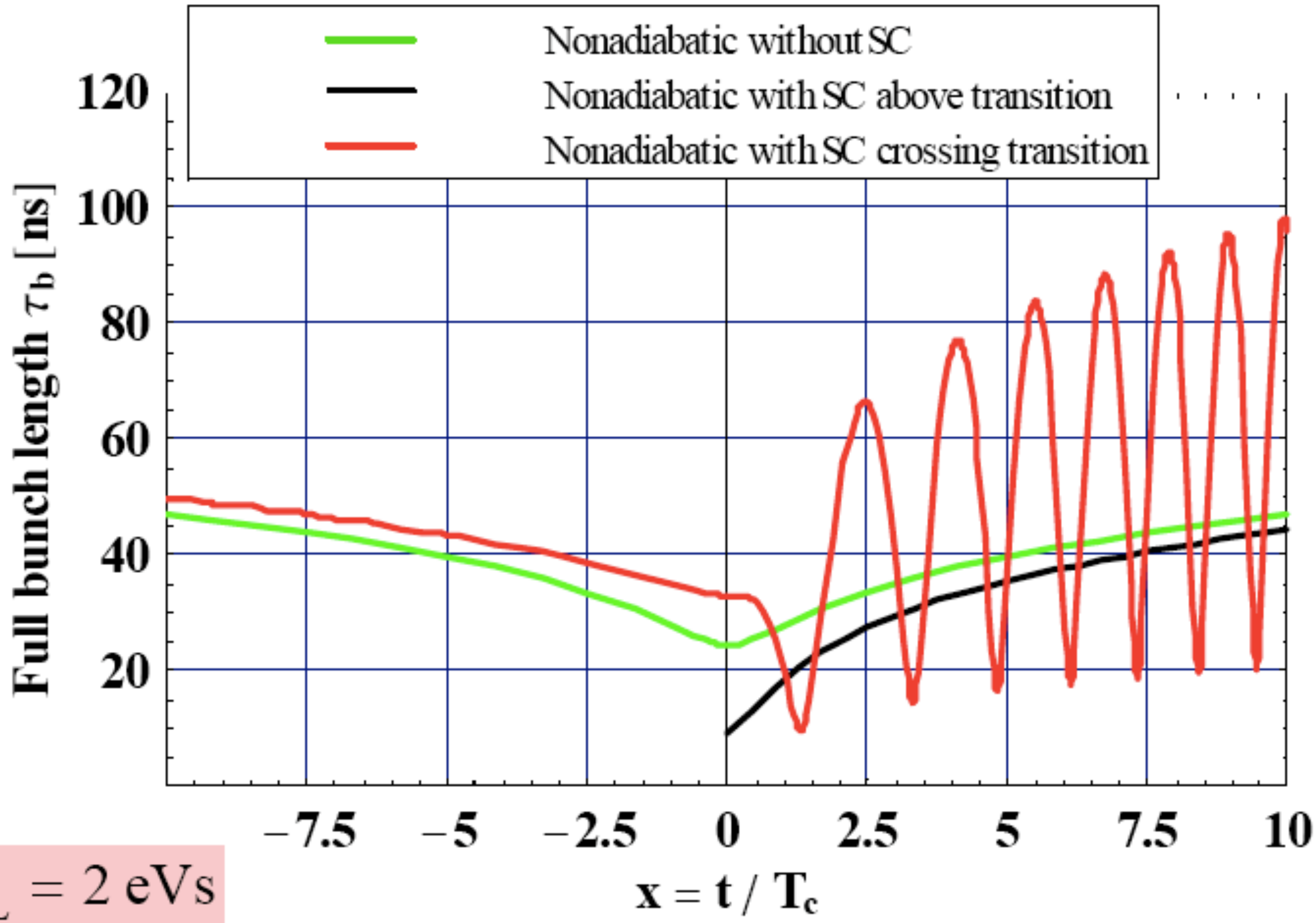
LONGITUDINAL ENVELOPE EQUATION (24/33)

STATIC (MATCHED) CASE



$\varepsilon_L = 2$ eVs

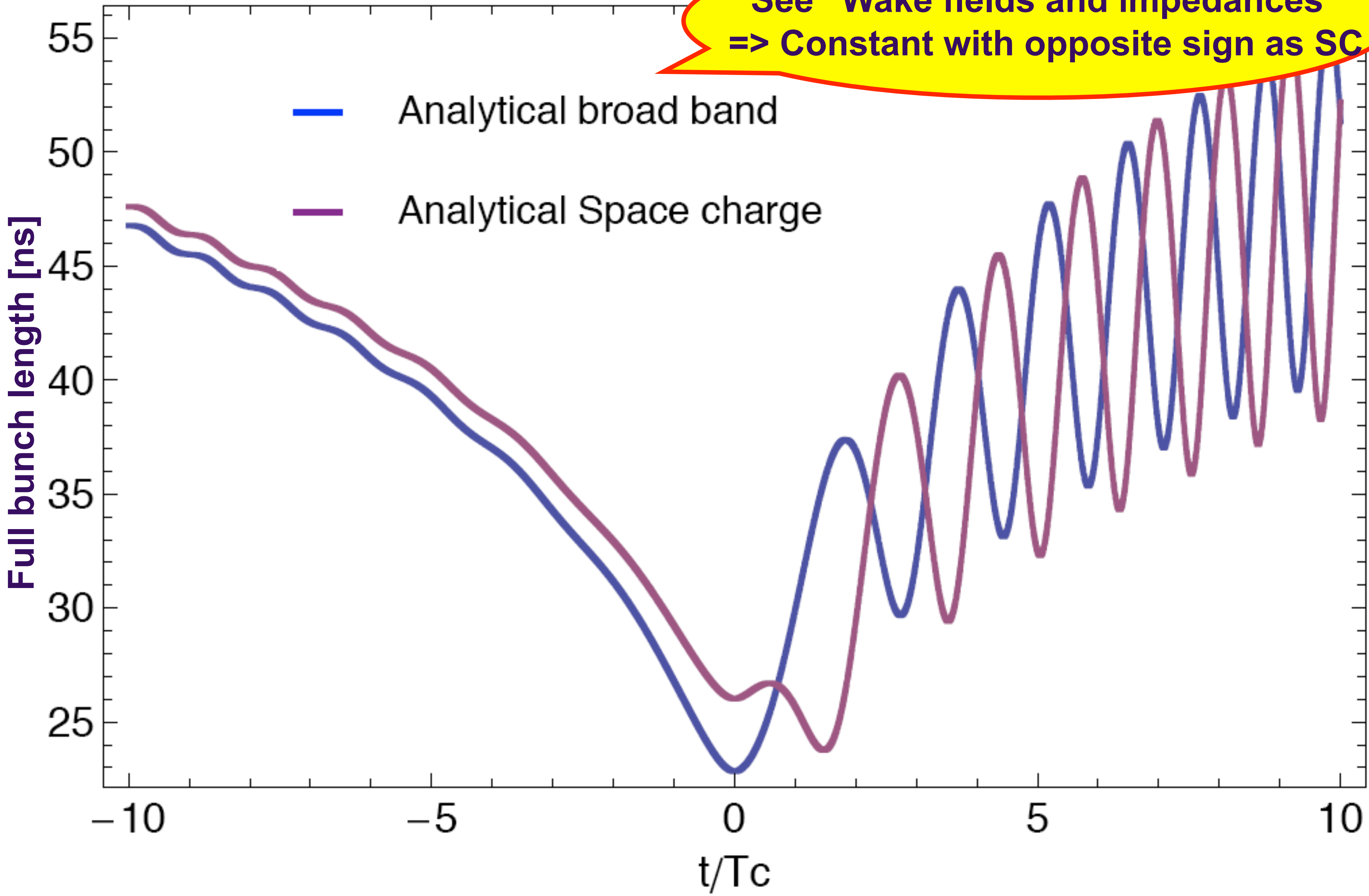
DYNAMIC ("REAL") CASE WHEN TRANSITION IS CROSSED



$\varepsilon_L = 2$ eVs

LONGITUDINAL ENVELOPE EQUATION (26/33)

See "Wake fields and impedances"
=> Constant with opposite sign as SC



LONGITUDINAL ENVELOPE EQUATION (27/33)

- **Bunch rotation with SC (and or constant inductive impedance)**

  => The envelope equation can also be used to estimate, for a given compression factor, the needed RF voltage, the compression time, the evolution of the bunch length and the evolution of the momentum spread

- **The envelope equation can also be (sometimes) written**

$$\frac{d^2 r_z}{d z^2} + K_{RF}\, r_z - \frac{K_{sc,l2}}{r_z^2} - \frac{\varepsilon_{l2}^2}{r_z^3} = 0$$

Half bunch length (in meters)

$$K_{RF} = \frac{e\,\hat{V}_{RF}\,h\,|\eta\cos\phi_s|}{2\pi R^2\,\gamma\,\beta^2\,m_0\,c^2}$$

$$K_{sc,l2} = \frac{3\,\hat{g}\,r_p\,N_b\,|\eta|}{2\,\gamma^3\beta^2}$$

Depends on the source (was 1 before)

$$\hat{g} = \frac{1}{2} + 2\ln\left(\frac{r_{pipe}}{r_{beam}}\right)$$

$$\varepsilon_{l2} = |\eta|\,r_{z0}\,\frac{\Delta p}{p_0} = cte$$

LONGITUDINAL ENVELOPE EQUATION (28/33)

- **The bunch is matched (at constant energy) when** $r_z = r_{z0}$ **=>** $\frac{d^2 r_z}{d z^2} = \frac{d r_z}{d z} = 0$

- **Before, we used** $\Delta\hat{\phi}$ **and we had** $\Delta\hat{\phi} = \pi \, h \, f_0 \, \tau_b$

Full bunch length [in s]

- **Here, we use** r_z **and we have** $r_z = \beta \, c \, \frac{\tau_b}{2}$

- **It can be checked that this equation is the same as the one we used before (with the small difference on the g-factor), remembering that**

Used before

$$\varepsilon_l = \frac{\pi}{2} \beta^2 \, \gamma \, E_{rest} \, \tau_b \, \frac{\Delta p}{p_0}$$

LONGITUDINAL ENVELOPE EQUATION (29/33)

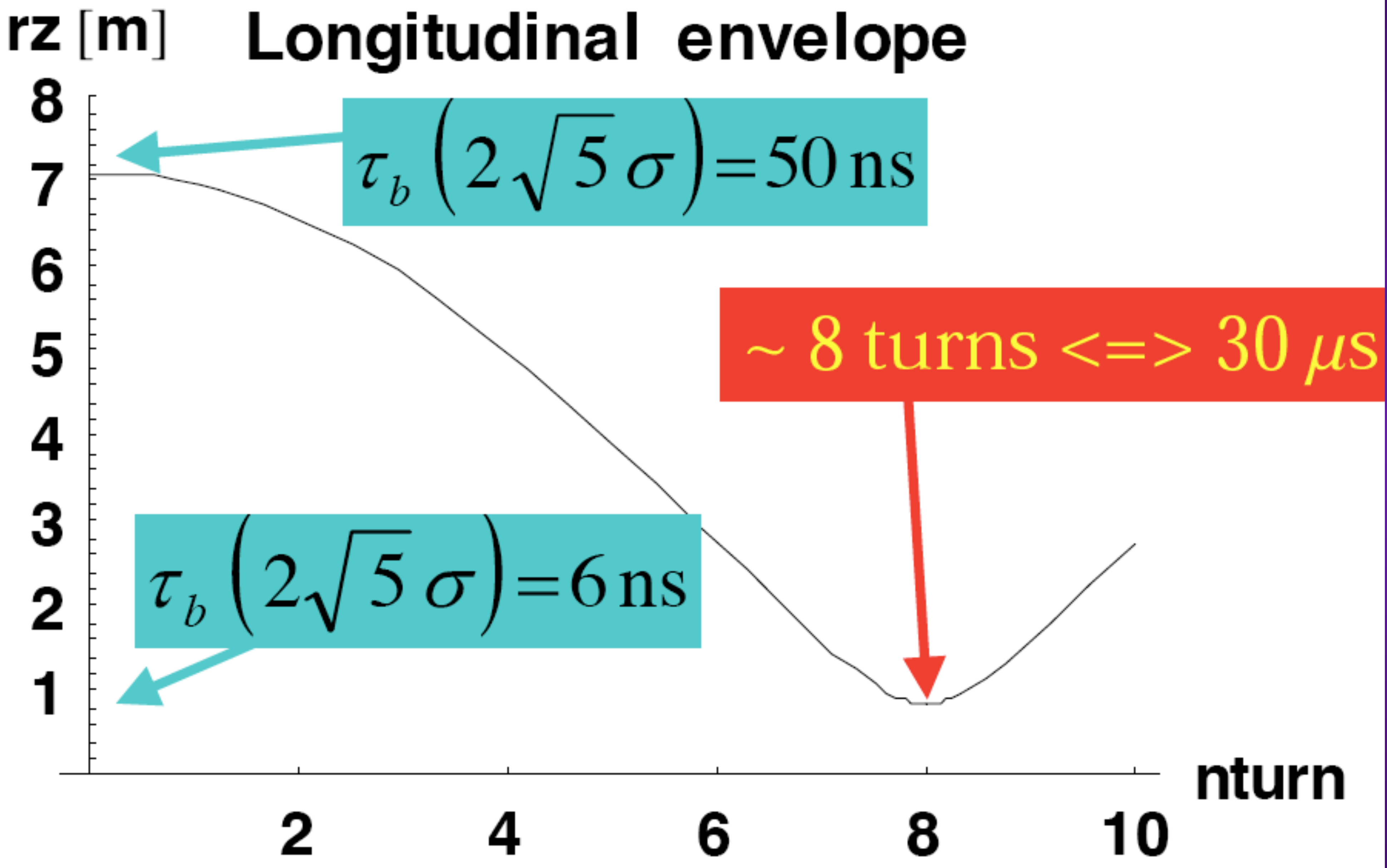
- **Example of bunch rotation estimated in the past in a 2 GeV proton-accumulator-compressor for a neutrino factory**

$\hat{V}_{RF} = 7\ \mathrm{MV}$ $h = 24$ $N_b = 10^{13}\ \mathrm{p/b}$

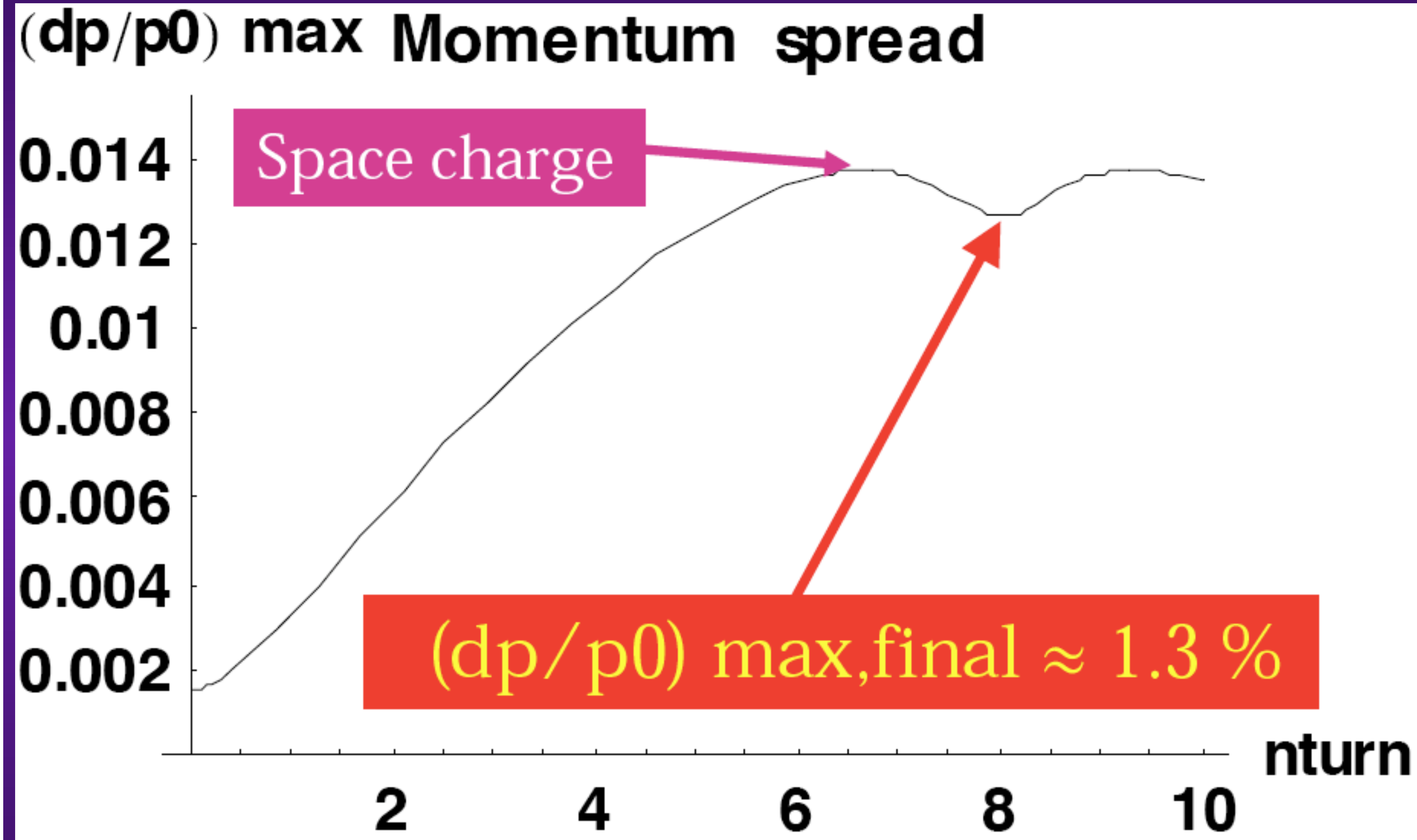
$\tau_b = 50\,\mathrm{ns}$ $\left(\Delta p / p_0\right)_{\max} = 2\left(\sigma_p / p_0\right) = 1.5 \times 10^{-3}$

$R = 150\ \mathrm{m}$ $\eta = -0.1$ $\hat{g} \approx 4$

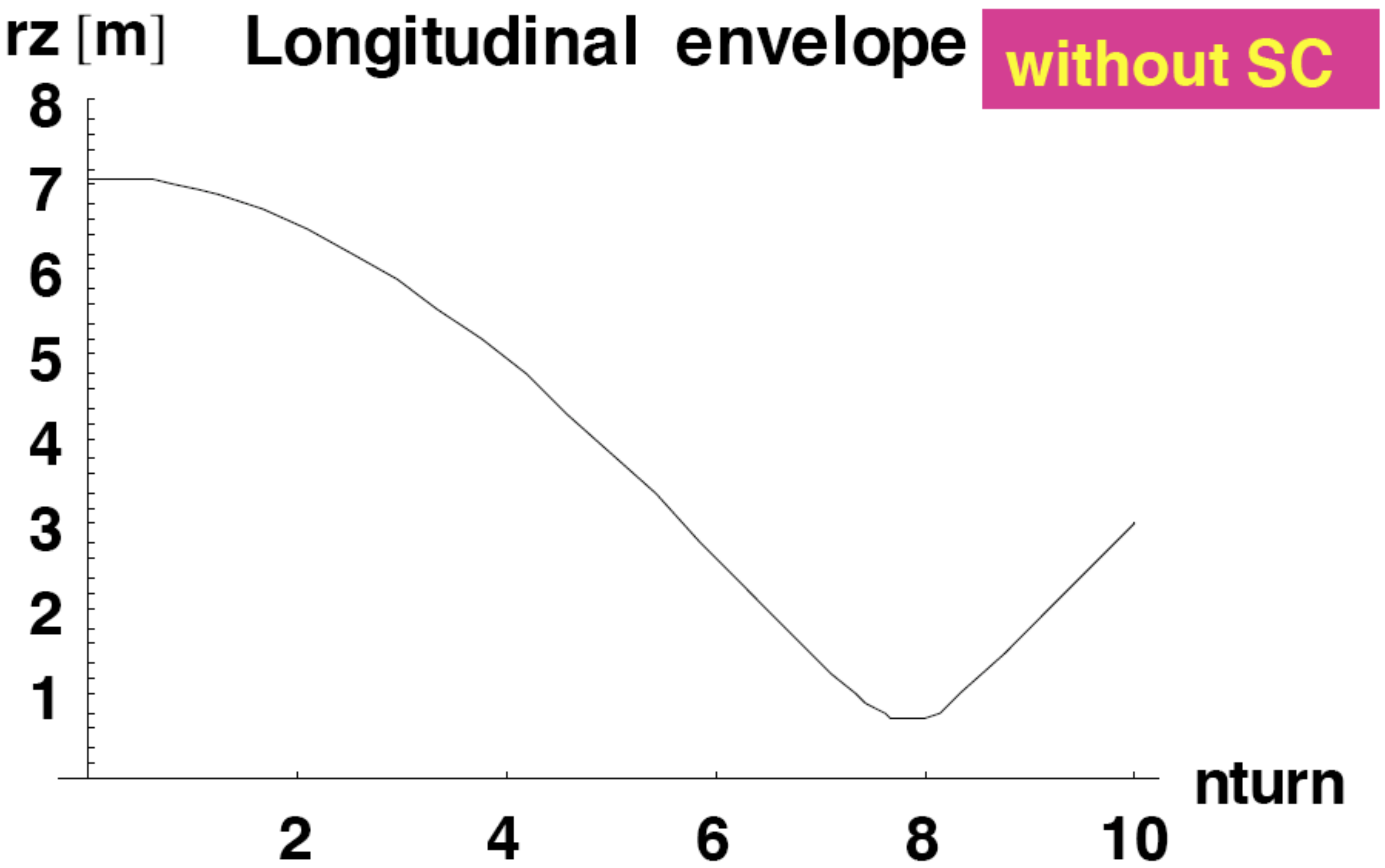
LONGITUDINAL ENVELOPE EQUATION (30/33)



LONGITUDINAL ENVELOPE EQUATION (31/33)



LONGITUDINAL ENVELOPE EQUATION (32/33)



LONGITUDINAL ENVELOPE EQUATION (33/33)