COASTING-BEAM TRANSVERSE COHERENT INSTABILITIES

- Single-particle equation formalism (9 Slides)
- Landau damping considering an externally given beam frequency spectrum (1)
- Physical origin of Landau damping (11)
- Landau damping of collective instabilities (9)
- Landau damping by external (1-D) non-linearity (17)
- General dispersion relation to be solved (7)
  - With momentum spread from chromaticity and/or slip factor
  - With 1D nonlinearity (in the plane of coherent motion)
- Landau damping for a collimated beam (17)
- Landau damping with 2-dimensional betatron tune spread from both octupoles and non-linear space charge (7)
- Effect of linear coupling between the transverse planes (35)
  - Transfer of instability growth rates
  - Transfer of Landau damping
  - Emittance sharing and exchange

SINGLE-PARTICLE EQUATION FORMALISM (1/9)

- **Single-particle (*i*) equation of motion (e.g. in the horizontal plane)**

$$\ddot{x}_i + Q_{0x,i}^2 \, \Omega_i^2 \, x_i = \frac{F_{x,i}}{\gamma \, m_0}$$

It was called Q_{x0} before

- **The (perturbative) force can be expanded to first order in terms of the test particle's motion and the average beam position to give**

$$F_{x,i} = \left(\frac{\partial F_{x,i}}{\partial x_i} \right)_{\bar{x}=0} x_i + \left(\frac{\partial F_{x,i}}{\partial \bar{x}} \right)_{x_i=0} \bar{x}$$

=> $Q_{x,i}^2 \approx Q_{0x,i}^2 + 2 \, Q_{0x,i} \, \Delta Q_{inc,x}$ **with** $\Delta Q_{inc,x} = - \frac{1}{2 \, Q_{0x,i} \, \Omega_i^2 \, \gamma \, m_0} \left(\frac{\partial F_{x,i}}{\partial x_i} \right)_{\bar{x}=0}$

It was called ΔQ_{incoh}^{x} before

SINGLE-PARTICLE EQUATION FORMALISM (2/9)

- The coherent motion can be solved by choosing $x_i = \overline{x}$

Contribution from quadrupolar impedance in case of asymmetric structure

Contribution from dipolar impedance

$$\Delta Q_{coh,x} = -\frac{1}{2\, Q_{0x,i}\, \Omega_i^2\, \gamma\, m_0} \left[\left(\frac{\partial F_{x,i}}{\partial x_i} \right)_{\overline{x}=0} + \left(\frac{\partial F_{x,i}}{\partial \overline{x}} \right)_{x_i=0} \right]$$

- Taking into account the external focusing forces, the coherent and incoherent forces (space charge + wall), the equation of motion of a test particle *i* can thus be written as

$$\ddot{x}_i + \Omega_i^2 \left(Q_{0x,i}^2 + 2\, Q_{0x,i}\, \Delta Q_{inc,x} \right) x_i = -2\, \Omega_i^2\, Q_{0x,i} \left(\Delta Q_{coh,x} - \Delta Q_{inc,x} \right) \overline{x}$$

- The relation between the "generalized Laslett tune shifts" and the horizontal coupling impedance is given by

Circular machine

$$Z_x(\omega) = \frac{j}{e\, \beta\, I\, \overline{x}} \int_0^{2\pi R} \overline{F}_{x,i}\, ds = -j\, \frac{2\pi R\, \gamma\, m_0}{e\, \beta\, I}\, 2\, \Omega_i^2\, Q_{0x,i} \left(\Delta Q_{coh,x} - \Delta Q_{inc,x} \right)_{(\omega)}$$

$$I = N\, e\, f_0$$

SINGLE-PARTICLE EQUATION FORMALISM (3/9)

- **In an accelerator, the spread in betatron frequency of the beam comes from several sources**

Betatron amplitudes

$$\Omega_i\left(p_i\right) = \Omega_0\left(1 - \eta\,\frac{\Delta p_i}{p_0}\right)$$

$$\Delta p_i = p_i - p_0$$

$$Q_{0x,i}\left(\hat{x}_i, \hat{y}_i, p_i\right) = Q_{0x0}\left(1 + \xi_x\,\frac{\Delta p_i}{p_0}\right) + f_{\text{ext}}\left(\hat{x}_i, \hat{y}_i\right)$$

$$\Delta Q_{inc,x}\left(\hat{x}_i, \hat{y}_i, p_i\right) = \Delta Q_{inc,x0} + \frac{\partial\,\Delta Q_{inc,x}}{\partial\,p_i}\,\Delta p_i + f_{\text{int}}\left(\hat{x}_i, \hat{y}_i\right)$$

$$\eta = \frac{1}{\gamma_{tr}^2} - \frac{1}{\gamma^2} = -\frac{d\,\Omega_i}{dp_i}\,\frac{p_i}{\Omega_i}$$

$$\xi_x = \frac{d\,Q_{0x,i}}{dp_i}\,\frac{p_i}{Q_{0x,i}}$$

SINGLE-PARTICLE EQUATION FORMALISM (4/9)

- The solution of the equation of unperturbed motion is written as

$$\ddot{x}_i + \dot{\varphi}_{x,i}^2 \, x_i = 0$$

=>

$$x_i = \hat{x}_i \cos\left(\varphi_{x,i} \right)$$

$$\dot{\varphi}_{x,i} = Q_{x,i}\,\Omega_i = \omega_{x,i} = Q_{x0}\,\Omega_0 \left(1 - \dot{\tau}_i \right) + \omega_{\xi_x}\,\dot{\tau}_i + \dot{\varphi}_{x,i}\left(\hat{x}_i , \hat{y}_i \right)$$

Constant => Coasting beam

$$\tau_i = \tau_0 + \dot{\tau}_i \, t$$

Time interval between the passage of the synchronous particle and the test particle

$$\dot{\tau}_i = \eta \, \frac{\Delta p_i}{p_i}$$

$$\omega_{\xi_x} = Q_{x0}\,\Omega_0 \, \frac{\xi_x}{\eta}$$

SINGLE-PARTICLE EQUATION FORMALISM (5/9)

=> $$\boxed{\varphi_{x,i} = \left[Q_{x0}\,\Omega_0 + \dot{\varphi}_{x,i}\left(\hat{x}_i, \hat{y}_i \right) \right] t + \left(\omega_{\xi_x} - Q_{x0}\,\Omega_0 \right) \tau_i + \varphi_{0x,i}}$$

- Suppose that at time $t = 0$ a perturbation is imposed on the beam so that each azimuthal slice is displaced transversally by an amount $\overline{x}\left(t = 0, \vartheta \right)$. This pattern is necessarily closed around the circumference and therefore can be decomposed into a Fourier series

$$\overline{x}\left(t = 0, \vartheta \right) = \sum_{n_x = -\infty}^{n_x = +\infty} \overline{X}_{n_x} \, e^{-j \, n_x \, \vartheta}$$

Azimuthal mode number.
The horizontal wave number is
$k_x = n_x / R$

=> It is only necessary to consider the evolution of a single sinusoidal wave n_x

$$\bar{x}\left(t=0,\vartheta\right) = \bar{X}\, e^{-j\, n_x\, \vartheta}$$

$$\bar{X} = \bar{X}_{n_x}$$

- The betatron oscillation of the beam centre which was at azimuth $\vartheta_{c,0}$ at time $t=0$ is then given by

$$\bar{x}\left(t,\vartheta_{c,0}\right) = \bar{X}\, e^{j\left(\omega_c\, t - n_x\, \vartheta_{c,0}\right)}$$

Coherent betatron frequency

$$\omega_c = Q_c\, \Omega_0$$

$$\vartheta = \vartheta_{c,0} + \Omega_0\, t$$

=> The position of the whole beam in azimuth and time can be described as follows

$$\bar{x}\left(t,\vartheta\right) = \bar{X}\, e^{j\left[\left(n_x + Q_c\right)\Omega_0\, t - n_x\, \vartheta\right]}$$

SINGLE-PARTICLE EQUATION FORMALISM (7/9)

- **Relation between the coherent tune and the local collective frequency (frequency observed at a fixed azimuth)**

$$\boxed{\omega = \left(n_x + Q_c \right) \Omega_0}$$

- **The influence of the wake fields can be calculated either in the laboratory or in a moving-frame that goes around with the particle. In the present case, the second (hydrodynamic) view is adopted. Then, the derivatives have to be taken along the orbit of the particle**

$$\boxed{d / dt = \partial / \partial t + \dot{\vartheta}_i \, \partial / \partial \vartheta_i = j \left(\omega - n_x \Omega_i \right)}$$

- **The steady-state solution of the test particle has the same time and azimuthal dependence as the driving term. We then look for a particular solution of the form**

$$x_i\left(t, \vartheta \right) = X_i \, e^{j\left(\omega t - n_x \vartheta \right)}$$

SINGLE-PARTICLE EQUATION FORMALISM (8/9)

=> $$X_i = -\frac{j e \beta I Z_x}{2 \pi R \gamma m_0} \left[\frac{\overline{X}}{\omega_{x,i}^2 - \left(\omega - n_x \Omega_i \right)^2} \right]$$

$$\omega_{x,i}^2 - \left(\omega - n_x \Omega_i \right)^2 \approx 2 \omega_{x0} \left[\omega_{x,i} - \left(\omega - n_x \Omega_i \right) \right]$$

$$X_i = -\frac{j e \beta I Z_x}{2 \omega_{x0} 2 \pi R \gamma m_0} \left[\frac{\overline{X}}{\omega_{x,i} - \left(\omega - n_x \Omega_i \right)} \right]$$

- Therefore, in the absence of a frequency spread => $X_i = \overline{X}$

$$\boxed{\Delta \omega_c = \omega_c - \omega_{x0} = U_x - j V_x}$$

Real frequency shift

Instability growth rate

SINGLE-PARTICLE EQUATION FORMALISM (9/9)

$$\left(U_x - jV_x\right)_{(\omega)} = \frac{j\, e\, \beta\, I\, Z_x(\omega)}{2\omega_{x0}\, 2\pi R\, \gamma\, m_0} = \frac{I\, c\, j\, Z_x(\omega)}{4\,\pi\, Q_{x0}\left(E_t / e\right)}$$

Laslett, Neil and Sessler (LNS) coefficents

$$\omega_c = \omega_R + j\,\omega_i$$

=>

$$e^{\,j\omega_c t} = e^{\,j(\omega_R + j\omega_i)\,t} = e^{\,j\omega_R t}\, e^{\frac{t}{\tau}}$$

- The instability rise-time [in s] is given by

$$\tau_x = -\frac{1}{\omega_i} = \frac{1}{V_x}$$

=>

$$\tau_x = \frac{4\,\pi\, Q_{x0}\left(E_t / e\right)}{I\, c \times \left\{-\mathrm{Re}\left[Z_x(\omega)\right]\right\}}$$

$$\omega = \left(n_x + Q_c\right)\Omega_0 \approx \left(n_x + Q_{x0}\right)\Omega_0$$

Reminder: The real part of the impedance is < 0 for frequencies < 0

LANDAU DAMPING CONSIDERING AN EXTERNALLY GIVEN BEAM FREQUENCY SPECTRUM

$$\int_{-\infty}^{+\infty} \rho_x\left(\omega_{x,i}\right)\, d\omega_{x,i} = 1$$

$$\overline{X} = \int_{-\infty}^{+\infty} \rho_x\left(\omega_{x,i}\right) X_i \, d\omega_{x,i}$$

Assuming that they do not depend on the incoherent frequencies

$$\left(\int_{-\infty}^{+\infty} \frac{\rho_x\left(\omega_{x,i}\right)\, d\omega_{x,i}}{\omega_c - \omega_{x,i}} \right)^{-1} = U_x - j V_x$$

- This expression constitutes a "dispersion relation", since it specifies the dependence of the oscillation frequency ω on the wave number $k_x = n_x / R$

- If the driving frequency belongs to the betatron frequency spread, then the dispersion relation contains a singularity. This leads us to the important concept of Landau damping, which will be reviewed in the next section

PHYSICAL ORIGIN OF LANDAU DAMPING (1/11)

- Landau damping is a general process that arises when one considers a whole collection of particles or other systems which have a spectrum of resonant frequencies, and interact in some way
- In accelerators we are usually concerned with an interaction of a kind that may make the beam unstable (wake fields), and we want to find out whether or not (and how) the spread of resonant frequencies will stabilise it
- If the particles have a spread in their natural frequencies, the motion of the particles can lose its coherency
- In order to understand the physical origin of this effect, let us first consider a simple harmonic oscillator which oscillates in the x-direction with its natural frequency. Let this oscillator be driven, starting at time t = 0, by a sinusoidal force. The equation of motion is

$$\boxed{\ddot{x}_i + \omega_{x,i}^2 \, x_i = f \cos\left(\omega_c \, t\right)}$$

$$x_i(0) = 0 \qquad \dot{x}_i(0) = 0$$

PHYSICAL ORIGIN OF LANDAU DAMPING (2/11)

- The solution is

$$x_i\left(t>0\right)=-\frac{f}{\omega_c^2-\omega_{x,i}^2}\left[\cos\left(\omega_c\,t\right)-\cos\left(\omega_{x,i}\,t\right)\right]=\frac{f}{2\,\omega_{x0}}\,\sin\left(\omega_{x0}\,t\right)\,\frac{\sin\left[\left(\omega_c-\omega_{x,i}\right)\,t/2\right]}{\left(\omega_c-\omega_{x,i}\right)\;/2}$$

- Consider now an ensemble of oscillators (each oscillator represents a single particle in the beam) which do not interact with each other and have a spectrum of natural frequency $\omega_{x,i}$ with a distribution $\rho_x\left(\omega_{x,i}\right)$ normalised to unity. As assumed previously, the origin of the betatron frequency spread is not specified: an externally given beam frequency spectrum is supposed
- Now starting at time t = 0, subject this ensemble of particles to the driving force $f\cos\left(\omega_c\,t\right)$ with all particles starting with initial conditions $x_i(0)=0$ and $\dot{x}_i(0)=0$. We are interested in the ensemble average of the response, which is given by superposition by

$$\bar{x}\left(t\right)=-\frac{f}{2\,\omega_{x0}}\int_{-\infty}^{+\infty}\frac{1}{\omega_c-\omega_{x,i}}\left[\cos\left(\omega_c\,t\right)-\cos\left(\omega_{x,i}\,t\right)\right]\rho_x\left(\omega_{x,i}\right)\,d\omega_{x,i}$$

PHYSICAL ORIGIN OF LANDAU DAMPING (3/11)

- **Changing the variable** $u = \omega_{x,i} - \omega_c$

$$\bar{x}(t) = \frac{f}{2\omega_{x0}} \int_{-\infty}^{+\infty} \left[\cos(\omega_c t) - \cos(\omega_c t + u t) \right] \frac{\rho_x(u + \omega_c)}{u} \, du$$

$$= \frac{f}{2\omega_{x0}} \left[\cos(\omega_c t) \int_{-\infty}^{+\infty} \left[1 - \cos(u t) \right] \frac{\rho_x(u + \omega_c)}{u} \, du + \sin(\omega_c t) \int_{-\infty}^{+\infty} \sin(u t) \frac{\rho_x(u + \omega_c)}{u} \, du \right]$$

- **If we are not interested in the transient effects immediately following the onset of the driving force, we may use the following formulae**

$$\lim_{t \to +\infty} \frac{\sin(u t)}{u} = \pi \, \delta(u) \qquad \lim_{t \to +\infty} \frac{1 - \cos(u t)}{u} = \mathrm{P.V.}\left(\frac{1}{u}\right)$$

PHYSICAL ORIGIN OF LANDAU DAMPING (4/11)

=>

$$\bar{x}(t) = \frac{f}{2\omega_{x0}} \left[\cos(\omega_c\, t)\, \mathrm{P.V.} \int_{-\infty}^{+\infty} \frac{\rho_x(\omega_{x,i})}{\omega_{x,i} - \omega_c}\, d\omega_{x,i} + \pi\, \rho_x(\omega_c)\, \sin(\omega_c\, t) \right]$$

- **The sign of the $\cos(\omega_c\, t)$ term relative to the driving force depends on the sign of $\mathrm{P.V.} \int_{-\infty}^{+\infty} d\omega_{x,i}\, \rho_x(\omega_{x,i}) / (\omega_{x,i} - \omega_c)$. Generally, this term is approximately given by $1/(\omega_{x,i} - \omega_c)$ outside the spectrum and crosses through zero somewhere inside the spectrum. A system is referred to as "capacitive" or "inductive" based on whether its sign is positive or negative**
- **The $\sin(\omega_c\, t)$ term has a definite sign relative to the driving force, because $\rho_x(\omega_c)$ is always positive. In particular, $\dot{\bar{x}}$ is always in phase with the force, indicating work is being done on the system. The system always reacts to the force "resistively"**

PHYSICAL ORIGIN OF LANDAU DAMPING (5/11)

- The Landau damping effect is to be distinguish from a "decoherence (also called phase-mixing, or filamentation) effect" that occurs when the beam has nonzero initial conditions. Had we included an initial condition $x_i(0) = x_{i,0}$ and $\dot{x}_i(0) = \dot{x}_{i,0}$, we would have introduced two additional terms into the ensemble response, which do not participate in the dynamic interaction of the beam particles and are not interesting for our purposes here. In this decoherence effect, individual particles continue to execute oscillations of constant amplitude, but the total beam response $\bar{x}$ decreases with time
- As mentioned above, work is continuously being done on the system. However, the amplitude of $\bar{x}$, as given before, does not increase with time. Where did the energy go?
- To investigate this, let us identify the energy of a particle as the square of its oscillation amplitude. The amplitude of the particle *i* is given by the slowly varying envelope

$$\mathrm{A}_{x,i} = \frac{f}{\omega_{x0}\left(\omega_c - \omega_{x,i}\right)} \sin\left[\frac{\left(\omega_c - \omega_{x,i}\right) t}{2}\right]$$

PHYSICAL ORIGIN OF LANDAU DAMPING (6/11)

- **This leads to a total oscillation energy of (with *N* the total number of particles in the beam)**

$$\mathrm{E}_x = N \int_{-\infty}^{+\infty} \rho_x\left(\omega_{x,i}\right) d\omega_{x,i} \left\{ \frac{f}{\omega_{x0}\left(\omega_c - \omega_{x,i}\right)} \sin\left[\frac{\left(\omega_c - \omega_{x,i}\right) t}{2} \right] \right\}^2$$

$$= \frac{N f^2}{\omega_{x0}^2} \int_{-\infty}^{+\infty} \rho_x\left(u + \omega_c\right) du \, \frac{\sin^2\left(u\, t / 2\right)}{u^2}$$

$$\lim_{t \to +\infty} \frac{\sin^2\left(u\, t/2\right)}{u^2} = \frac{\pi\, t}{2} \delta\left(u\right)$$

=>

$$\boxed{\mathrm{E}_x = \frac{\pi N f^2}{2 \omega_{x0}^2} \rho_x\left(\omega_c\right) t}$$

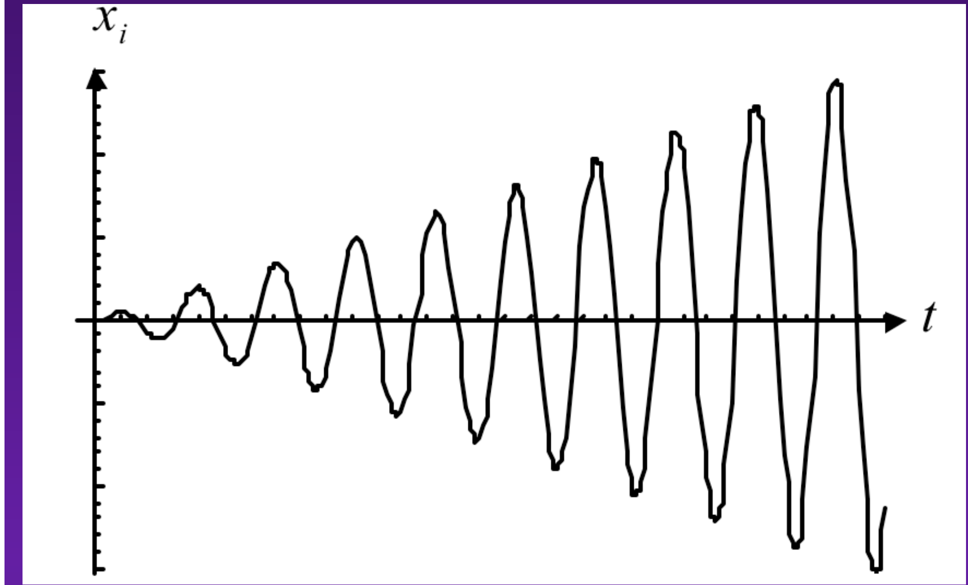
=> The system therefore absorbs energy from the driving force indefinitely while holding the ensemble beam response within bounds

PHYSICAL ORIGIN OF LANDAU DAMPING (7/11)

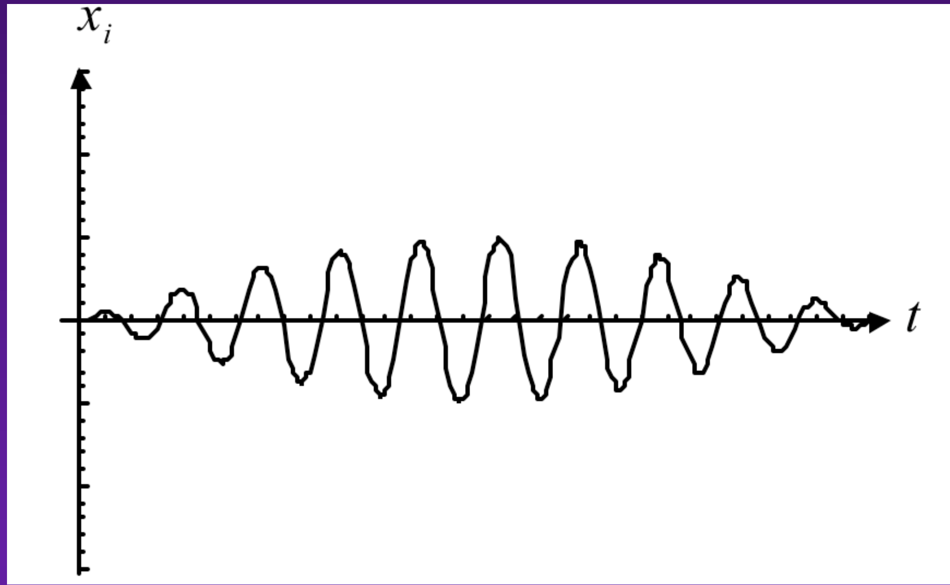
- **The stored energy is incoherent in the sense that the energy is contained in the individual particles, but it is not to be regarded as heat in the system. This is because the stored energy is not distributed more or less uniformly in all particles, but is selectively stored in particles with continuously narrowing range of frequencies around the driving frequency**

- **Observe 2 particles $\omega_{x,i} = \omega_c$ and $\omega_{x,i} \neq \omega_c$. At the beginning, they oscillate "coherently" (same amplitude and same phase). However, after a while the particle with $\omega_{x,i} = \omega_c$, being resonantly driven, continues to increase in amplitude as time increases, whereas the other particle with $\omega_{x,i} \neq \omega_c$ realizes that its frequency is not the same as the driving one and the "beating" phenomenon is observed for this particle**

PHYSICAL ORIGIN OF LANDAU DAMPING (8/11)



$$\omega_{x,i} = \omega_c$$



$$\omega_{x,i} \neq \omega_c$$

PHYSICAL ORIGIN OF LANDAU DAMPING (9/11)

- The particle with $\omega_{x,i} \neq \omega_c$ is considered off-coherence when $\sin\left[\left(\omega_c - \omega_{x,i}\right) t / 2\right]$ is maximum, i.e. after the time approximately $\bar{t} = \pi / \left|\omega_c - \omega_{x,i}\right|$, and at time $2\bar{t}$, the particle returns all its energy back to the driving force in a beating process. One can also say that, if one considers the phenomenon for a time t, only the particles with $\left|\omega_c - \omega_{x,i}\right| < \pi / t$ still oscillate coherently. All the others are "beating". It is mainly those particles with $\left|\omega_c - \omega_{x,i}\right| < \pi / t$ that contribute to the $\sin\left(\omega_c\, t\right)$ response, and those particles with $\left|\omega_c - \omega_{x,i}\right| > \pi / t$ that contribute to the $\cos\left(\omega_c\, t\right)$ response. Since the number of particles with $\left|\omega_c - \omega_{x,i}\right| < \pi / t$ decreases with time as $1/t$ while their amplitude increases as t, the net $\sin\left(\omega_c\, t\right)$ contribution to $\bar{x}$ is constant with time

PHYSICAL ORIGIN OF LANDAU DAMPING (10/11)

- **A mathematical trick bypasses most of these subtleties and makes the analysis much more concise**
- **Using complex notation, the driving force is written $f\, e^{j\omega_c t}$ and the single-particle motion gives the total beam response**

$$\overline{x} = -\frac{f\, e^{j\omega_c t}}{2\omega_{x0}} \int_{-\infty}^{+\infty} \frac{1}{\omega_c - \omega_{x,i}}\, \rho_x\left(\omega_{x,i}\right)\, d\omega_{x,i}$$

- **If one considers the integration along the real axis of the $\omega_{x,i}$ − plane , but moves the pole at $\omega_{x,i} = \omega_c$ down by an infinitesimal amount**

$$\boxed{\omega_c \rightarrow \omega_c - j\,\varepsilon}$$

PHYSICAL ORIGIN OF LANDAU DAMPING (11/11)

=> $$\boxed{\overline{x} = \frac{f\, e^{j\,\omega_c\, t}}{2\,\omega_{x0}} \left[-\,\mathrm{P.V.} \int_{-\infty}^{+\infty} \frac{\rho_x\left(\omega_{x,i}\right)}{\omega_c - \omega_{x,i}}\, d\omega_{x,i} - j\,\pi\, \rho_x\left(\omega_c\right) \right]}$$

- **Introducing an infinitesimally small imaginary part to the coherent frequency has the physical meaning of considering a force that has a time dependence of** $e^{j\,\omega_c\, t + \varepsilon\, t}$ **, i.e. a force that grows with time at an infinitesimal rate. This means the driving force has not been in existence since , which has the same effect as introducing explicit initial conditions as far as removing the singularity is concerned**
- **Note that the term in brackets is called the "Beam Transfer function", i.e. it is the beam response to a sinusoidal driving force (it is in fact the 0-intensity limit of the BTF). It can be determined experimentally by measuring the phase and amplitude of the beam response. The interest of the BTF is that it contains information about the beam and the accelerator => Can be used to measure the beam frequency spectrum and the accelerator impedance**

LANDAU DAMPING OF COLLECTIVE INSTABILITIES (1/9)

- **Results obtained in the previous section, when applied to circular accelerators, lead to Landau damping of collective instabilities**
- **Reminder: In the case of a coasting-beam, the following equation has to be solved**

$$\left(\int_{-\infty}^{+\infty} \frac{\rho_x\left(\omega_{x,i}\right)\, d\omega_{x,i}}{\omega_c - \omega_{x,i}} \right)^{-1} = U_x - j V_x$$

Contains information about the beam frequency spectrum

Contains information about the beam intensity and impedance

$$Q_c\,\Omega_i = \omega_c = Q_c\,\Omega_0$$

$$\Delta\omega_c = \omega_c - \omega_{x0} = U_x - j V_x = \frac{I\, c\, j\, Z_x\left(\omega\right)}{4\,\pi\, Q_{x0}\left(E_t / e\right)}$$

Stability $\Leftrightarrow$ $V_x \leq 0$

LANDAU DAMPING OF COLLECTIVE INSTABILITIES (2/9)

- Consider the real parameter $\omega_c - \omega_{x0}$ (stability limit) and observes the locus traced out in the complex $D - \text{plane}$ as $\omega_c - \omega_{x0}$ is scanned form $-\infty$ to $+\infty$

$$D = \left(\int_{-\infty}^{+\infty} \frac{\rho_x\left(\omega_{x,i}\right) \, d\omega_{x,i}}{\omega_c - \omega_{x,i}} \right)^{-1}$$

Note that the low-intensity BTF is equal to 1 / D

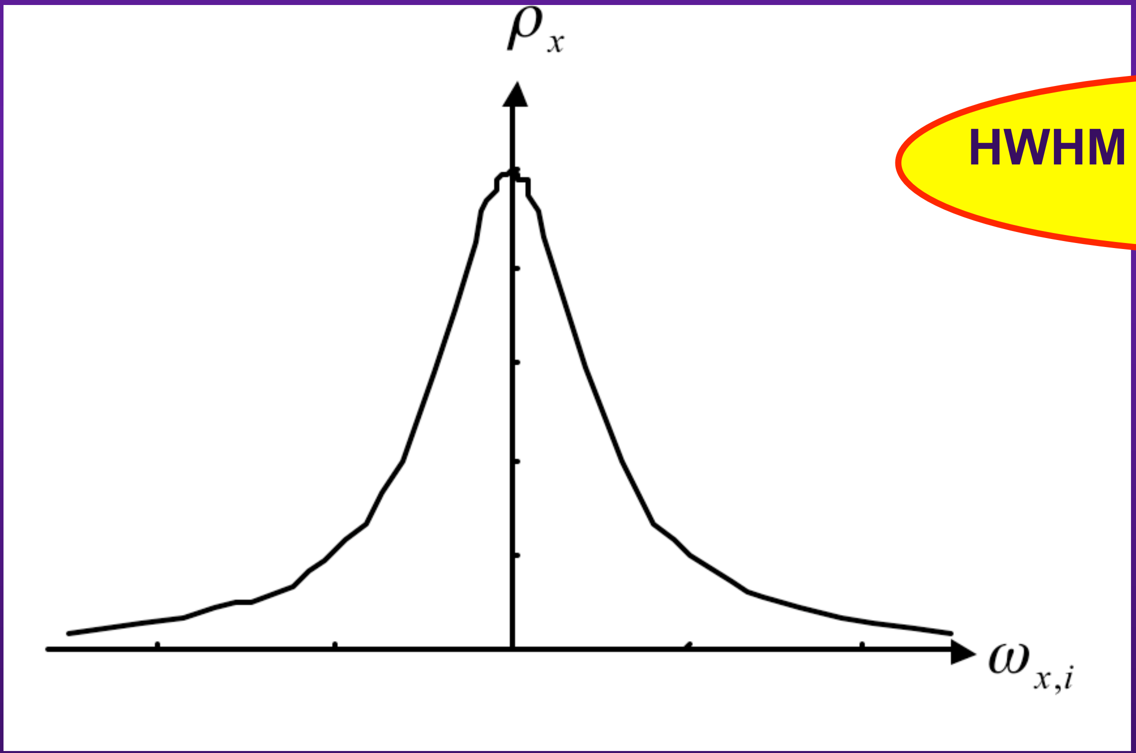
- This locus defines a "stability boundary diagram". The r.h.s (complex frequency shift in the absence of frequency spread), is then plotted as a single point. If this point lies on the boundary, it means the solution of the dispersion equation is real, and it is such that the beam is just at the edge of instability. If it lies on the inside of the stability diagram (the side which contains the origin of the $D - \text{plane}$), the beam is stable. If it lies on the outside of it, the beam is unstable

LANDAU DAMPING OF COLLECTIVE INSTABILITIES (3/9)

- The dispersion relation is particularly simple for the **Lorentzian spectrum**. In this case the distribution function is written

$$\boxed{\rho_x\left(\omega_{x,i}\right)=\frac{\delta\omega_x}{\pi}\left[\left(\omega_{x,i}-\omega_{x0}\right)^2+\delta\omega_x^2\right]^{-1}}$$

HWHM = Half Width at Half Maximum



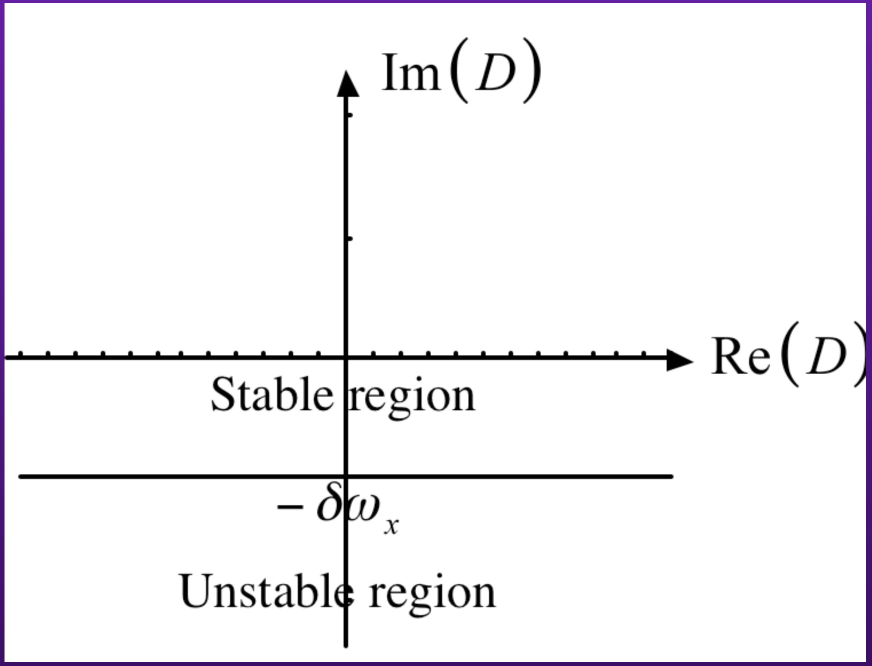
LANDAU DAMPING OF COLLECTIVE INSTABILITIES (4/9)

- **The corresponding dispersion integral is given by**

$$\boxed{\int_{-\infty}^{+\infty} \frac{\rho_x\left(\omega_{x,i}\right) d\omega_{x,i}}{\omega_c - \omega_{x,i}} = \frac{1}{\omega_c - \omega_{x0} - j\,\delta\omega_x}}$$

=> $$\omega_c = \omega_{x0} + U_x + j\left(\delta\omega_x - V_x\right)$$

Stability ⇔ $$\boxed{\delta\omega_x \geq V_x}$$



LANDAU DAMPING OF COLLECTIVE INSTABILITIES (5/9)

- The larger the frequency spread, the stronger the Landau damping. Also, for a given spread, the effectiveness of Landau damping is different for different spectral shapes. The Lorentzian spectrum, having a long distribution tail, is very effective; spectra with cutoff tails tend to be less effective, while the spectrum, of course, is not effective at all. Sharp edges in a spectral shape are reflected in sharp edges in the stability boundary

- The Lorentzian distribution describes the stabilizing mechanism of Landau damping in a simple way, but neglects an important point. The real part of the complex mode frequency shift is not taken into account in the stability criterion because of its infinite tails. However, realistic distributions have finite tails and for distributions without important tails, Landau damping is prevented when the shift is larger than the frequency spread. This is explained by the large detuning which shifts the coherent frequency to a value outside the spectrum . This kills Landau damping since they are no individual particles which can couple to the coherent response

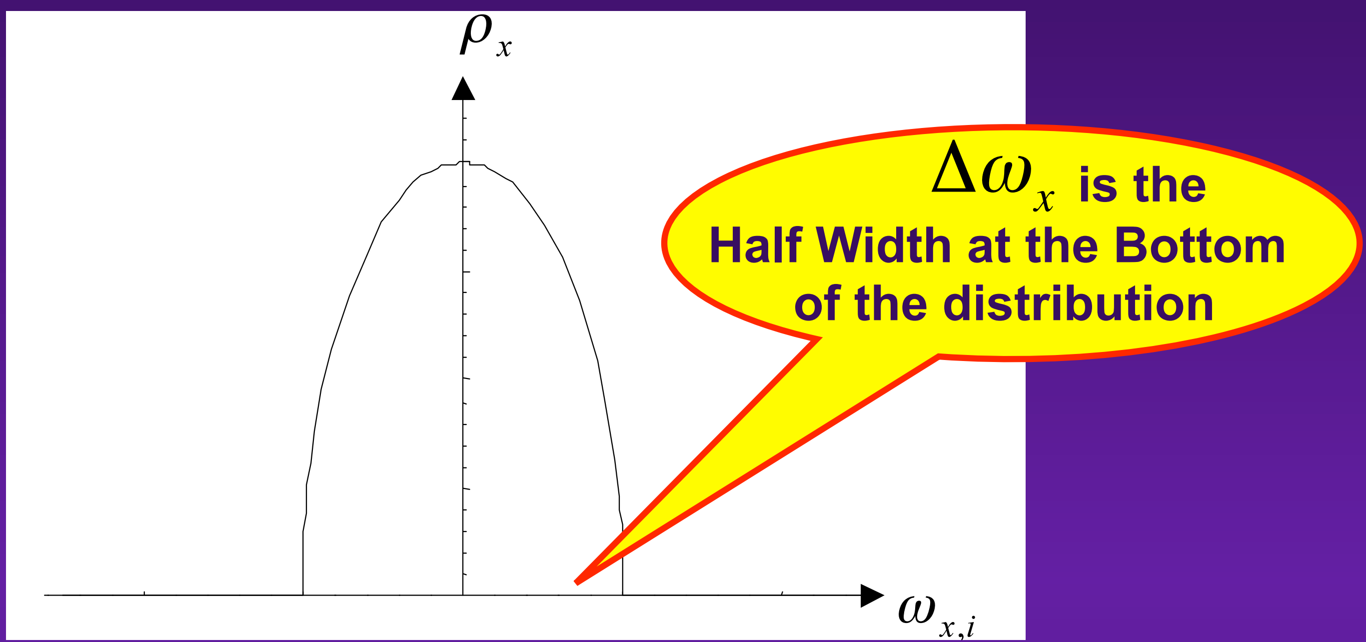
LANDAU DAMPING OF COLLECTIVE INSTABILITIES (6/9)

- For practical accelerator operations, there may be approximate information on the value of the frequency spread, but not enough detailed information on the shape of the frequency spectrum; or there may be only a need of a rough estimate of whether the collective instability is Landau damped. For those purposes, a simplified stability criterion derived using the elliptical spectrum is introduced, knowing that Lorentzian and elliptical spectra are limiting cases and that realistic distributions are probably between them

- The dispersion relation in the case of an elliptical distribution is given by

$$\boxed{\rho_x\left(\omega_{x,i}\right) = \begin{cases} \dfrac{2}{\pi \Delta\omega_x^2}\sqrt{\Delta\omega_x^2 - \left(\omega_{x,i} - \omega_{x0}\right)^2} & , \quad \left|\omega_{x,i} - \omega_{x0}\right| \le \Delta\omega_x \\ 0 & , \quad \left|\omega_{x,i} - \omega_{x0}\right| > \Delta\omega_x \end{cases}}$$

LANDAU DAMPING OF COLLECTIVE INSTABILITIES (7/9)



- **The corresponding dispersion integral is given by**

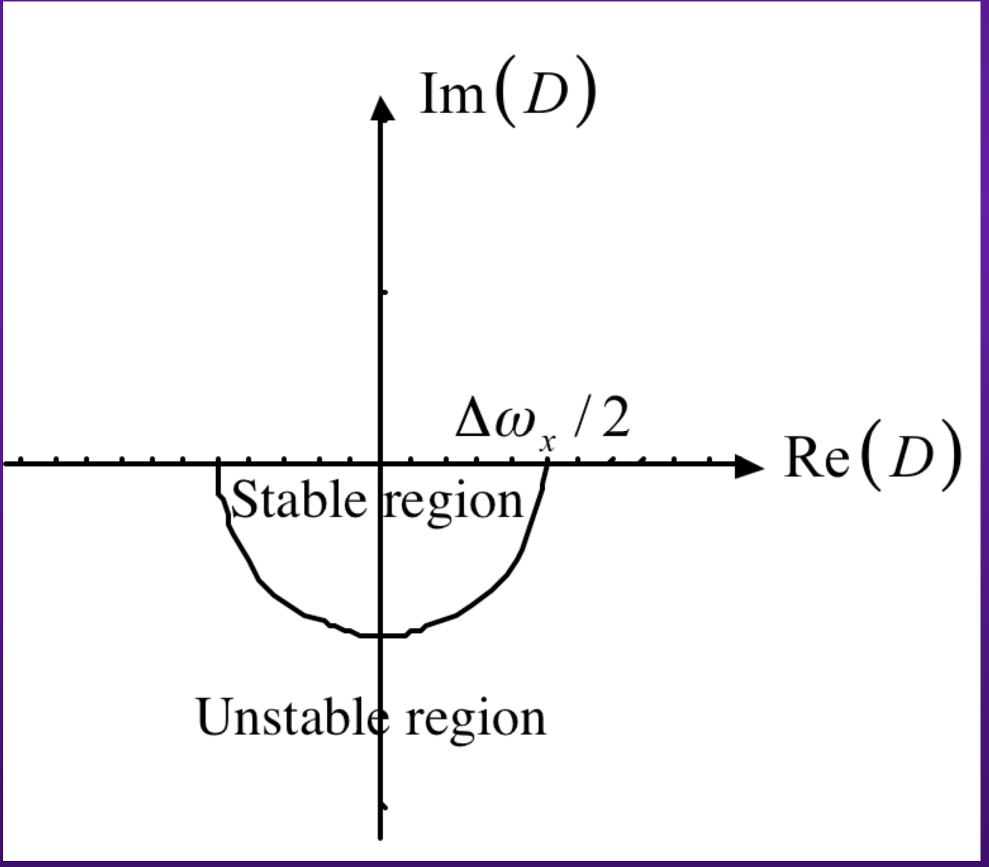
$$\int_{-\infty}^{+\infty} \frac{\rho_x\left(\omega_{x,i}\right)\, d\omega_{x,i}}{\omega_c - \omega_{x,i}} = 2\left[\omega_c - \omega_{x0} - j\sqrt{\Delta\omega_x^2 - \left(\omega_c - \omega_{x0}\right)^2}\,\right]^{-1}$$

LANDAU DAMPING OF COLLECTIVE INSTABILITIES (8/9)

- **This leads to**

$$\omega_c = \omega_{x0} + U_x \frac{\Delta\omega_x^2 + 4\left(U_x^2 + V_x^2\right)}{4\left(U_x^2 + V_x^2\right)} + j V_x \frac{\Delta\omega_x^2 - 4\left(U_x^2 + V_x^2\right)}{4\left(U_x^2 + V_x^2\right)}$$

Stability ⇔ $\boxed{\Delta\omega_x \geq 2\left|U_x - j V_x\right|}$



LANDAU DAMPING OF COLLECTIVE INSTABILITIES (9/9)

- **A fair comparison of two spectral shapes can be made when they have the same Half Width at Half Height (HWHH):**
  - **For the Lorentzian spectrum:** $\Delta\omega_{\mathrm{HWHH}}^{x} = \delta\omega_x$ $\quad \Delta\omega_{\mathrm{HWHH}}^{x} \geq V_x$
  - **For the elliptical spectrum:** $\Delta\omega_{\mathrm{HWHH}}^{x} = \left(\sqrt{3}/2\right)\Delta\omega_x$

$$\boxed{\Delta\omega_{\mathrm{HWHH}}^{x} \geq \sqrt{3}\,\left|U_x - j\,V_x\right|}$$

- **Although the exact stability condition depends on details of the spectrum, the last equation is an important result which takes into account both contributions of the real and imaginary parts of the coherent tune shift => It says that if the real mode frequency shift or growth rate, calculated without Landau damping, is larger than**

$$\Delta\omega_{\mathrm{HWHH}}^{x} / \sqrt{3}$$

Landau damping most likely will not rescue the beam from instability

LANDAU DAMPING BY EXTERNAL NONLINEARITY (1/17)

- The origins of the frequency spread that leads to Landau damping have been specified at the beginning but have not been taken into account till now. The case where the frequency spread comes from the longitudinal momentum spread of the beam is straightforward (for a coasting beam), because the longitudinal momentum is a constant, which just affects the coefficients in the equations of motion of the transverse oscillations, and hence their frequencies. It can be dealt with the same method as in the previous sections. The same result applies also if one considers a tune spread that is due to a non-linearity in the other plane $Q_{x,i}\left(\hat{y}_i\right)$
- However, this result is no longer valid if the non-linearity is in the plane of coherent motion $Q_{x,i}\left(\hat{x}_i\right)$
- In this case, the steady-state is more involved because the coherent motion is then a small addition to the large incoherent amplitudes that make the frequency spread, and it is inconsistent to assume that it can be treated as a linear superposition => One needs to consider "second order" non-linear terms

LANDAU DAMPING BY EXTERNAL NONLINEARITY (2/17)

- **Let us thus proceed to include the frequency spread due to a non-linearity in the external focusing, in the plane of coherent motion only**
- **One will show that the steady-state response of**

$$\ddot{x}_i + \omega_{x,i}^2\left(\hat{x}_i\right) x_i = 2\,\omega_{x0}\left(-U_x + j\,V_x\right)\,\overline{x}$$

is more involved than the simple-minded response found before

$$x_i = 2\,\omega_{x0}\left(-U_x + j\,V_x\right)\left[\frac{\overline{x}}{\omega_{x,i}^2 - \left(\omega - n_x\,\Omega_0\right)^2}\right]$$

and is given by (in 1st order in $K_x = \left(d\omega_{x,i} / d\hat{x}_i\right) / \left(\omega_{x,i} / \hat{x}_i\right) << 1$ **)**

$$X_i = 2\,\omega_{x0}\left(-U_x + j\,V_x\right)\overline{X}\left[\left(1 - \frac{K_x}{2}\right)\frac{1}{\omega_{x,i}^2\left(\hat{x}_i\right) - \left(\omega - n_x\,\Omega_0\right)^2} - \frac{K_x}{2}\,\frac{\omega_{x,i}^2\left(\hat{x}_i\right) + \left(\omega - n_x\,\Omega_0\right)^2}{\left[\omega_{x,i}^2\left(\hat{x}_i\right) - \left(\omega - n_x\,\Omega_0\right)^2\right]^2}\right]$$

LANDAU DAMPING BY EXTERNAL NONLINEARITY (3/17)

- **To do so let's consider the following case (slightly different notations compared to the previous slides) of free oscillation**

$$\ddot{x} + \nu_0^2 \, x + F\left(x \right) = 0$$

$$F\left(0 \right) = F'\left(0 \right) = 0$$

Represents the nonlinear part of the restoring force

=> (see also the treatment of space-charge nonlinearities)

$$x = A \cos\left[\nu\left(A \right) t + \psi \right] + \text{higher harmonics}$$

- **As we need to know the response of a large-amplitude particle to a small external perturbation, we will try and solve**

$$\ddot{x} + \nu_0^2 \, x + F\left(x \right) = B \, e^{\, j \omega t}$$

LANDAU DAMPING BY EXTERNAL NONLINEARITY (4/17)

$$x_1 = x - x_0$$

with $$\ddot{x}_0 + \nu_0^2 \, x_0 + F\left(x_0\right) = 0$$

- We assume B small and drops terms higher than 1st order in it. The choice of the appropriate x_0 avoids B-independent part of x_1, which will be proportional to B

- Let's consider 1st the effect of a perturbation in the form of an impulse

$$\ddot{x} + \nu_0^2 \, x + F\left(x\right) = B \, \delta\left(t - t_0\right)$$

For $t < t_0$ $$x = x_0 = A \cos\left[\nu \, t + \psi\right]$$

$$\nu\left(A\right)$$

LANDAU DAMPING BY EXTERNAL NONLINEARITY (5/17)

For t > t_0

$$x = \left(A + \delta A \right) \cos\left[\nu\left(A + \delta A \right) t + \psi + \delta\psi \right]$$

Small constants to be determined (near t_0)

$$x\left(t = t_0^+ \right) - x\left(t = t_0^- \right) = 0$$

$$\dot{x}\left(t = t_0^+ \right) - \dot{x}\left(t = t_0^- \right) = B$$

- **Let's call** $$K = \frac{A \, d\nu}{\nu \, dA}$$

LANDAU DAMPING BY EXTERNAL NONLINEARITY (6/17)

- **Let us 1st show that (as it will be used to determine $\dot{x}$)**

$$\cos\left[\nu\left(A+\delta A\right)t+\psi\right]=\cos\left[\nu\left(A\right)t+\psi\right]-\delta\nu\;t\;\sin\left[\nu\left(A\right)t+\psi\right]$$

$$\frac{d}{dA}\left\{\cos\left[\nu\left(A\right)t+\psi\right]\right\}=-\frac{d\nu}{dA}\;t\;\sin\left[\nu\left(A\right)t+\psi\right]$$

And, by definition of a derivative

$$f'\left(x_0\right)=\lim_{x\to x_0}\frac{f\left(x\right)-\left(x_0\right)}{x-x_0}=\lim_{h\to 0}\frac{f\left(x_0+h\right)-f\left(x_0\right)}{h}$$

LANDAU DAMPING BY EXTERNAL NONLINEARITY (7/17)

=>
$$\frac{d}{dA}\left\{\cos\left[\nu\left(A\right)t+\psi\right]\right\}=\frac{\cos\left[\nu\left(A+\delta A\right)t+\psi\right]-\cos\left[\nu\left(A\right)t+\psi\right]}{\delta A}$$

- **For t > t_0**

$$x=\left(A+\delta A\right)\cos\left[\nu\left(A+\delta A\right)t+\psi+\delta\psi\right]$$
$$=\left(A+\delta A\right)\left\{\begin{array}{l}\cos\left[\nu\,t+\psi\right]-\delta\psi\,\sin\left[\nu\,t+\psi\right]\\ -\delta\nu\,t\,\sin\left[\nu\,t+\psi\right]-\delta\nu\,t\,\delta\psi\,\cos\left[\nu\,t+\psi\right]\end{array}\right\}$$

$$\dot{x}=\dot{x}_0-\delta A\,\nu\left(1+K\right)\sin\left(\nu\,t+\psi\right)-\delta A\,\nu^2\,t\,K\cos\left(\nu\,t+\psi\right)$$
$$-\,\delta\psi\,A\,\nu\cos\left(\nu\,t+\psi\right)$$

LANDAU DAMPING BY EXTERNAL NONLINEARITY (8/17)

- **Applying the 2 conditions near t_0 yields**

$$\left(A + \delta A \right) \cos\left[\nu\left(A + \delta A \right) t_0 + \psi + \delta\psi \right] = A \cos\left[\nu\left(A \right) t_0 + \psi \right]$$

$$B = -\,\delta A\, \nu \left(1 + K \right) \sin\left(\nu\, t_0 + \psi \right) - \delta A\, \nu^2\, t\, K \cos\left(\nu\, t_0 + \psi \right)$$
$$-\,\delta\psi\, A\, \nu \cos\left(\nu\, t_0 + \psi \right)$$

- **The solutions are**

$$\delta A = -\,B\, \frac{\sin\phi_0}{\nu \left(1 + K \sin^2\phi_0 \right)}$$

$$\delta\psi = B\, \frac{K\, \nu\, t_0 \sin\phi_0 - \cos\phi_0}{A\, \nu \left(1 + K \sin^2\phi_0 \right)}$$

$$\phi = \nu\, t + \psi$$

$$\phi_0 = \nu\, t_0 + \psi$$

LANDAU DAMPING BY EXTERNAL NONLINEARITY (9/17)

- **Therefore,**

For t < t_0 $$x_1^\delta = 0$$

For t > t_0 (averaging over ψ)

$$x_1^\delta = \frac{B}{\nu}\left(1 - \frac{K}{2}\right)\sin\left[\left(t - t_0\right)\nu\right] + \frac{B\,K}{2}\left(t - t_0\right)\cos\left[\left(t - t_0\right)\nu\right]$$

- **Finally, to solve the case with simple-harmonic perturbation, we must multiply by $e^{j\omega t_0}$ and integrate dt_0 from - ∞ to t**

$$x_1 = \int_{-\infty}^{t} x_1^\delta \, e^{j\omega t_0} \, dt_0$$

LANDAU DAMPING BY EXTERNAL NONLINEARITY (10/17)

=> This yields, after having assumed a negative imaginary part of ω (i.e. assuming an instability) which makes the oscillating terms for t -> - ∞ converge to 0

$$x_1 = \left(1 - \frac{K}{2} \right) \frac{B\, e^{j\omega t}}{\nu^2 - \omega^2} - \frac{K}{2} \frac{\left(\nu^2 + \omega^2 \right) B\, e^{j\omega t}}{\left(\nu^2 - \omega^2 \right)^2}$$

When K = 0, the usual response is recovered

$$x_1 \left(K = 0 \right) = \frac{B\, e^{j\omega t}}{\nu^2 - \omega^2}$$

LANDAU DAMPING BY EXTERNAL NONLINEARITY (11/17)

- **UNIFORM-DENSITY CASE**

$$x_0 = A \cos\left[\nu\, t + \psi\right]$$

$$\dot{x}_0 = -\,\nu\, A \cos\left[\nu\, t + \psi\right]$$

=> The ellipse in phase space has the area $\pi\, A^2\, \nu$

$$d\left(\pi\, A^2\, \nu\right) = \pi\left(2\, A\, \delta A\, \nu + A^2\, \delta\nu\right)$$

$$= 2\pi\left(1 + \frac{K}{2}\right)\nu\, A\, dA$$

LANDAU DAMPING BY EXTERNAL NONLINEARITY (12/17)

=> The response factor averaged over the uniformly filled area is

$$R_u = \left\langle \frac{x_1}{B\, e^{j \omega t}} \right\rangle = \frac{1}{\pi\, A_m^2\, \nu_m} \int_0^{A_m} \frac{x_1}{B\, e^{j \omega t}}\, d\left(\pi\, A^2\, \nu \right)$$

=>

$$R_u = \frac{1}{A_m^2\, \nu_m} \int_0^{A_m} 2\, A \left\{ \frac{\nu}{\nu^2 - \omega^2} - \frac{K}{2} \frac{\nu \left(\nu^2 + \omega^2 \right)}{\left(\nu^2 - \omega^2 \right)^2} \right\} dA$$

Furthermore, it can be shown that

$$2\, A \left\{ \frac{\nu}{\nu^2 - \omega^2} - \frac{K}{2} \frac{\nu \left(\nu^2 + \omega^2 \right)}{\left(\nu^2 - \omega^2 \right)^2} \right\} = \frac{d}{dA} \left(\frac{\nu\, A^2}{\nu^2 - \omega^2} \right)$$

LANDAU DAMPING BY EXTERNAL NONLINEARITY (13/17)

=> $$R_u = \frac{1}{A_m^2 \, \nu_m} \int_0^{A_m} \frac{d}{dA} \left(\frac{\nu \, A^2}{\nu^2 - \omega^2} \right) dA = \frac{1}{\nu^2 - \omega^2}$$

Therefore:

- **The uniformly filled region behaves as though all the particles had a resonant frequency corresponding to the one of the greatest amplitude**
- **Such a system has no Landau damping**

LANDAU DAMPING BY EXTERNAL NONLINEARITY (14/17)

- **NON-UNIFORM DENSITY**

$$N = \int_0^\infty \rho(A) \, d\left(\pi A^2 \nu\right)$$

Number of particles

=> The response factor averaged over the distribution is

$$R_{nu} = \left\langle \frac{x_1}{B \, e^{j \omega t}} \right\rangle = \frac{1}{N} \int_0^\infty \frac{x_1}{B \, e^{j \omega t}} \rho(A) \, d\left(\pi A^2 \nu\right)$$

$$= \frac{1}{N} \int_0^\infty \left\{ \frac{\nu}{\nu^2 - \omega^2} - \frac{K}{2} \frac{\nu \left(\nu^2 + \omega^2\right)}{\left(\nu^2 - \omega^2\right)^2} \right\} \rho(A) \, 2 \pi A \, dA$$

Integrating this by parts yields

$$R_{nu} = \frac{1}{N} \left[u \, v \right]_0^{\infty} - \frac{1}{N} \int_0^{\infty} u' \, v \, dA$$

with

$$A \left[\frac{\nu}{\nu^2 - \omega^2} - \frac{K}{2} \frac{\nu \left(\nu^2 + \omega^2 \right)}{\left(\nu^2 - \omega^2 \right)^2} \right] = v'$$

$$v = \frac{\nu \, A^2}{2 \left(\nu^2 - \omega^2 \right)}$$

$$2 \pi \rho \left(A \right) = u$$

$$u' = 2 \pi \rho' \left(A \right)$$

=>

$$R_{nu} = - \frac{\pi}{N} \int_0^{\infty} \rho' \left(A \right) \frac{\nu \, A^2}{\nu^2 - \omega^2} \, dA$$

Therefore: It is the derivative of the distribution function which matters and not the distribution itself => Regions of uniform density contribute nothing! The previous result is recovered

LANDAU DAMPING BY EXTERNAL NONLINEARITY (16/17)

- **Finally, coming back to our initial problem, i.e. to solve**

$$\ddot{x}_i + \omega_{x,i}^2(\hat{x}_i)\, x_i = 2\omega_{x0}(-U_x + jV_x)\,\overline{x}$$

The solution can be written

$$\left[-\frac{\pi}{N}\int_0^{+\infty} \rho'_x(\hat{x}_i)\, \frac{\omega_{x,i}(\hat{x}_i)\, \hat{x}_i^2}{\omega_{x,i}^2(\hat{x}_i) - (\omega - n_x \Omega_i)^2}\, d\hat{x}_i\right]^{-1} = 2\omega_{x0}(-U_x + jV_x)$$

or, using another distribution function (which will also be used later)

$$\boxed{\int_{\hat{x}_i=0}^{\hat{x}_i=+\infty} f_{x0}(\hat{x}_i)\, \hat{x}_i\, d\hat{x}_i = \frac{1}{2\pi}} \quad \Rightarrow \quad \int_{\hat{x}_i=0}^{\hat{x}_i=+\infty} \frac{\partial f_{x0}(\hat{x}_i)}{\partial \hat{x}_i}\, \hat{x}_i^{\,2}\, d\hat{x}_i = -\frac{1}{\pi}$$

LANDAU DAMPING BY EXTERNAL NONLINEARITY (17/17)

One finally obtains

$$\left[\int_{0}^{+\infty} \frac{ -\pi \dfrac{d f_{x0}\left(\hat{x}_i\right)}{d\hat{x}_i} \hat{x}_i^2 }{ \omega_c - \omega_{x,i}\left(\hat{x}_i\right) } d\hat{x}_i \right]^{-1} = U_x - j\,V_x$$

This result is also found using the Vlasov formalism (which can also be used for coasting beams) as we will see for bunches beams

GENERAL DISPERSION RELATION TO BE SOLVED (1/7)

- **If in addition, one considers also a momentum spread and a spread from nonlinearity of the second transverse planes (as it is the case for instance when octupoles are used), the dispersion equation writes**

$$\boxed{\left(\int_{\hat{x}_i=0}^{\hat{x}_i=+\infty} \int_{\hat{y}_i=0}^{\hat{y}_i=+\infty} \int_{p_i=0}^{p_i=+\infty} \frac{-2\pi^2 \dfrac{d f_{x0}(\hat{x}_i)}{d\hat{x}_i} \hat{x}_i^2 \, f_{y0}(\hat{y}_i) \, \hat{y}_i \, g_0(p_i)}{\omega - \left[n_x + Q_{x,i}(\hat{x}_i, \hat{y}_i, p_i) \right] \Omega_i(p_i)} \, d\hat{x}_i \, d\hat{y}_i \, dp_i \right)^{-1} = U_x - j \, V_x}$$

$$\int_{\hat{y}_i=0}^{\hat{y}_i=+\infty} f_{y0}(\hat{y}_i) \, \hat{y}_i \, d\hat{y}_i = \frac{1}{2\pi} \qquad \int_0^{+\infty} g_0(p_i) \, dp_i = 1$$

$$\omega - \left[n_x + Q_{x,i}(\hat{x}_i, \hat{y}_i, p_i) \right] \Omega_i(p_i) = \omega_c - \omega_{x0} + \left[(n_x + Q_{x0}) \, \Omega_0 - \omega_{\xi_x} \right] \dot{\tau}_i - \dot{\varphi}_{x,i}(\hat{x}_i, \hat{y}_i)$$

$$= \omega - (n_x + Q_{x0}) \, \Omega_0 + \boxed{\left[(n_x + Q_{x0}) \, \Omega_0 - \omega_{\xi_x} \right] \dot{\tau}_i - \dot{\varphi}_{x,i}(\hat{x}_i, \hat{y}_i)}$$

Betatron spectrum

Incoherent frequency spread

GENERAL DISPERSION RELATION TO BE SOLVED (2/7)

- Consider 1st the case with no nonlinearity, i.e. only a momentum spread (with a parabolic distribution)

$$\omega - \left[n_x + Q_{x,i}\left(\hat{x}_i, \hat{y}_i, p_i \right) \right] \Omega_i\left(p_i \right) = \omega_c - \omega_{x0} - \Delta\omega_p \, \frac{\dot{\tau}_i}{\dot{\tau}_L}$$

$$\dot{\tau}_L = \left| \eta \left(\frac{\Delta p}{p} \right)_L \right|$$

Positive spread (as we look at negative frequencies)

$$\Delta\omega_p = - \left[\left(n_x + Q_{x0} \right) \Omega_0 - \omega_{\xi_x} \right] \dot{\tau}_L$$

Note that even at transition there is a spread through the chromaticity

$$\left(\int_{\dot{\tau}_i = -\dot{\tau}_L}^{\dot{\tau}_i = +\dot{\tau}_L} \frac{\frac{2\pi}{\Omega_0} g_0\left(\dot{\tau}_i \right)}{\omega_c - \omega_{x0} - \Delta\omega_p \frac{\dot{\tau}_i}{\dot{\tau}_L}} \, d\dot{\tau}_i \right)^{-1} = U_x - j \, V_x$$

The chromatic frequency has to be > 0 to avoid loss of spread

$$\omega_{\xi_x} = Q_{x0} \, \Omega_0 \, \frac{\xi_x}{\eta}$$

$$g_0\left(\dot{\tau}_i \right) = \frac{3\Omega_0}{8\pi \dot{\tau}_L} \left[1 - \left(\frac{\dot{\tau}_i}{\dot{\tau}_L} \right)^2 \right]$$

$$\int_{\dot{\tau}_i = -\dot{\tau}_L}^{\dot{\tau}_i = +\dot{\tau}_L} g_0\left(\dot{\tau}_i \right) d\dot{\tau}_i = \frac{\Omega_0}{2\pi}$$

GENERAL DISPERSION RELATION TO BE SOLVED (3/7)

=> $$\left(\int_{-1}^{+1} \frac{1 - x^2}{x_1 - x} \, dx \right)^{-1} = \left(U_x - j\, V_x \right) \times \frac{3}{4\, \Delta\omega_p}$$

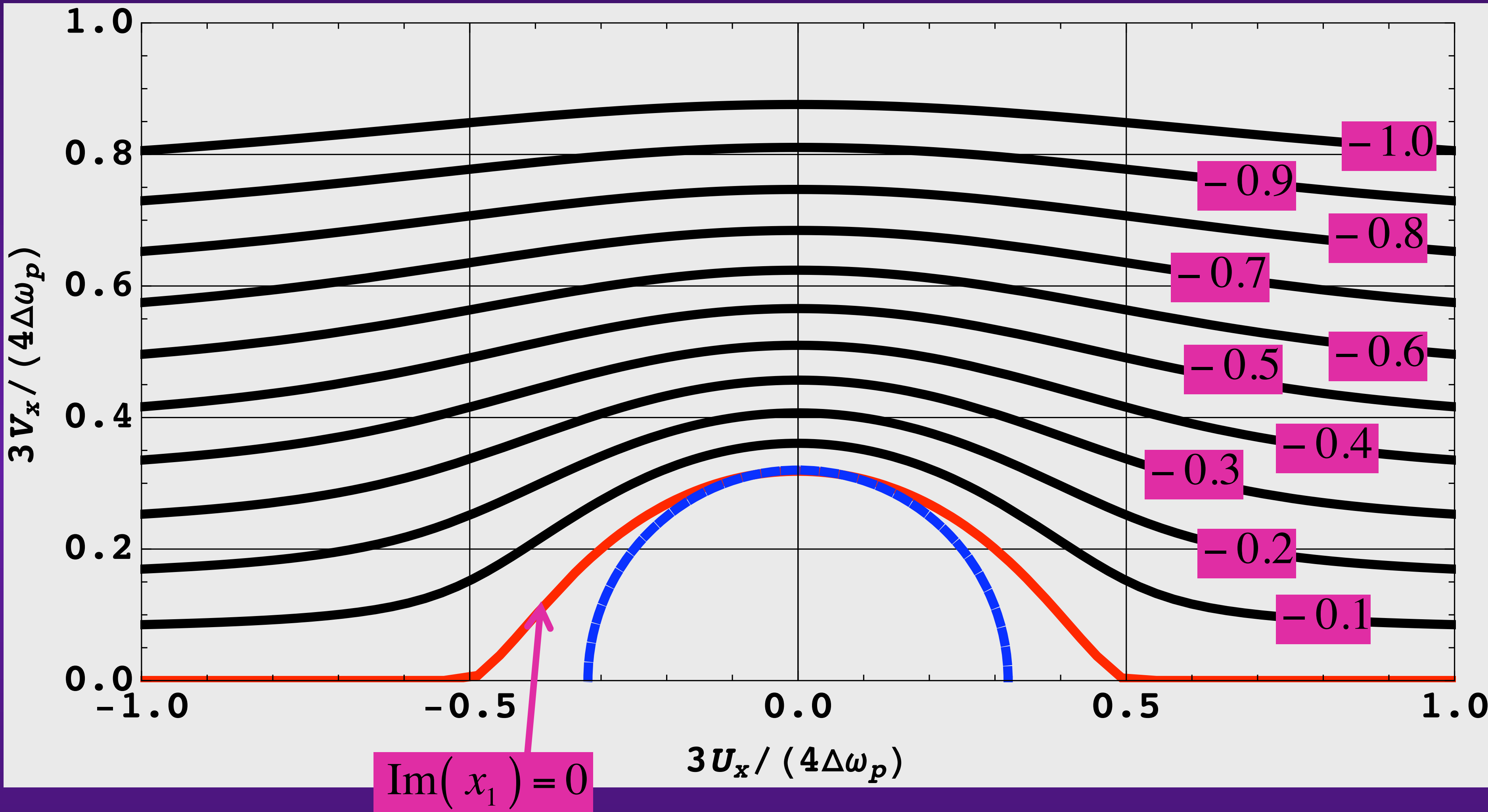
$$x = \frac{\dot{\tau}_i}{\dot{\tau}_L} \qquad x_1 = \frac{\omega_c - \omega_{x0}}{\Delta\omega_p} = \frac{\Delta\omega_c}{\Delta\omega_p}$$

Looking at the stability diagram of the next slide a stability criterion (circle approximation) can be obtained

$$\left| U_x - j\, V_x \right| \times \frac{3}{4\, \Delta\omega_p} \leq 0.3$$ => $$\boxed{\Delta\omega_p \geq 2.5 \left| U_x - j\, V_x \right|}$$

Stability criterion we found before with an elliptical spectrum (with 2 instead of 2.5)

GENERAL DISPERSION RELATION TO BE SOLVED (4/7)



$$\tau = -\frac{1}{\mathrm{Im}(x_1)\,\Delta\omega_p}$$

GENERAL DISPERSION RELATION TO BE SOLVED (5/7)

- **Consider now the case with a nonlinearity only in the horizontal plane (i.e. in the plane of coherent motion), and no momentum spread**

$$\omega - \left[n_x + Q_{x,i}\left(\hat{x}_i, \hat{y}_i, p_i\right) \right] \Omega_i\left(p_i\right) = \omega_c - \omega_{x0} - \dot{\varphi}_{x,i}\left(\hat{x}_i, \hat{y}_i\right)$$

$$\dot{\varphi}_{x,i}\left(\hat{x}_i, \hat{y}_i\right) = \frac{\partial Q_{x,i}}{\partial \hat{x}_i^2} \hat{x}_i^2 \, \Omega_0 = \Delta\omega_{nl} \frac{\hat{x}_i^2}{\hat{x}_L^2}$$

$$\left(\int_{\hat{x}_i = 0}^{\hat{x}_L} \frac{-\pi \dfrac{d f_{x0}\left(\hat{x}_i\right)}{d\hat{x}_i} \hat{x}_i^2}{\omega_c - \omega_{x0} - \Delta\omega_{nl} \dfrac{\hat{x}_i^2}{\hat{x}_L^2}} \, d\hat{x}_i \right)^{-1} = U_x - j\, V_x$$

$$f_{x0}\left(\hat{x}_i\right) = \frac{2}{\pi \hat{x}_L^2} \left[1 - \left(\frac{\hat{x}_i}{\hat{x}_L} \right)^2 \right]$$

GENERAL DISPERSION RELATION TO BE SOLVED (6/7)

=> $$\left(\int_0^1 \frac{x}{x_1 - x} \, dx \right)^{-1} = \left(U_x - j \, V_x \right) \times \frac{2}{\Delta\omega_{nl}}$$

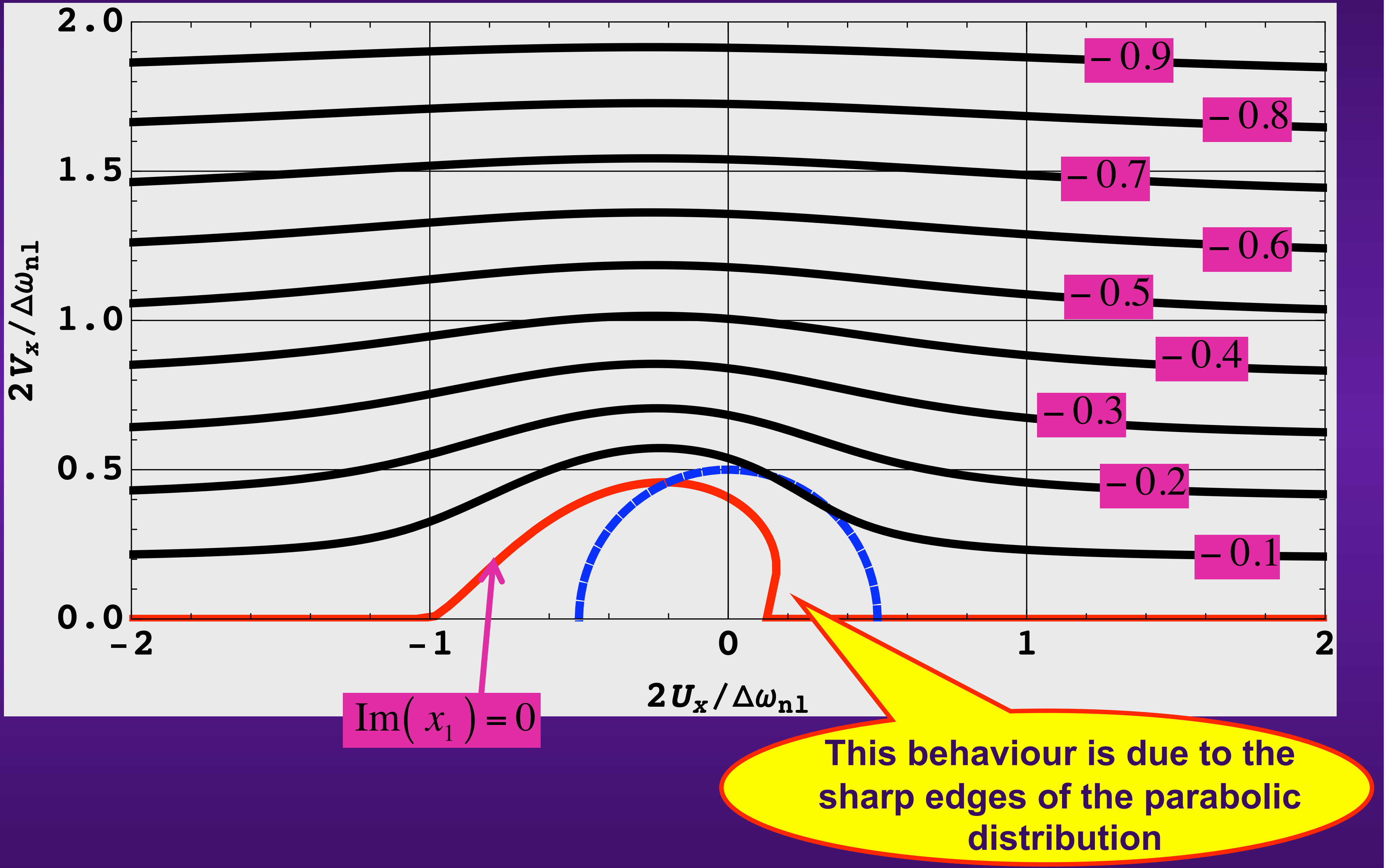
$$x = \left(\frac{\hat{x}_i}{\hat{x}_L} \right)^2 \qquad x_1 = \frac{\omega_c - \omega_{x0}}{\Delta\omega_{nl}} = \frac{\Delta\omega_c}{\Delta\omega_{nl}}$$

Looking at the stability diagram of the next slide a stability criterion (circle approximation, knowing that this stability diagram exhibits pathologies linked to the sharp edges of the parabolic distribution) can be obtained

$$\left| U_x - j \, V_x \right| \times \frac{2}{\Delta\omega_{nl}} \leq 0.5$$ => $$\boxed{\Delta\omega_{nl} \geq 4 \left| U_x - j \, V_x \right|}$$

Stability criterion we found before with an elliptical spectrum (with 2 instead of 4) => Usually a fair comparison is done by comparing the Half Width at Half Maximum

GENERAL DISPERSION RELATION TO BE SOLVED (7/7)



LANDAU DAMPING FOR A BEAM COLLIMATED AT AN ARBITRARY NUMBER OF SIGMAS (1/17)

- **The Landau damping mechanism heavily depends on the tails of the distribution. To look at it let's consider the following case, neglecting the momentum spread (and incoherent tune shift), but treating "correctly" the two-dimensional betatron tune spread introduced by (Landau) octupoles**

  => The dispersion relation to be solved (for a round beam) is

$$
1 = - \, \Delta Q_{coh}^{x} \int_{J_x = 0}^{+\infty} dJ_x \int_{J_y = 0}^{+\infty} dJ_y \, \frac{J_x \dfrac{\partial f\left(J_x, J_y\right)}{\partial J_x}}{Q_c - Q_x\left(J_x, J_y\right) - m \, Q_s}
$$

= 0 for coasting beams

- **Here, the action-angle variables are used**

$$
q_{x,y} = \sqrt{2 \, J_{x,y}} \, \cos\vartheta_{x,y} \qquad p_{x,y} = \sqrt{2 \, J_{x,y}} \, \sin\vartheta_{x,y} \qquad J_{x,y} = \frac{q_{x,y}^2 + p_{x,y}^2}{2}
$$

$$
\sigma_{x,y} = \sqrt{\varepsilon}
$$

Normalised rms beam sizes

$$
< J_{x,y} > = \varepsilon \qquad H_{x,y}^0 = \omega \, J_{x,y}
$$

LANDAU DAMPING FOR A BEAM COLLIMATED AT AN ARBITRARY NUMBER OF SIGMAS (2/17)

- **In the presence of octupoles, the transverse tunes are given by**

$$\boxed{Q_x\left(J_x, J_y\right) = Q_0 + a_0 \, J_x + b_0 \, J_y}$$

$$a_0 = \frac{3}{8\,\pi} \int \beta_x^2 \, \frac{O_3}{B\rho} \, ds \qquad b_0 = -\frac{3}{8\,\pi} \int 2\,\beta_x \, \beta_y \, \frac{O_3}{B\rho} \, ds$$

$$B_y = O_3 \left(x^3 - 3\,x\,y^2 \right) \qquad K_3 \, [\mathrm{m}^{-4}] = \frac{1}{B\,\rho} \times \frac{\partial^3 B_y}{\partial \, x^3} = 6 \, O_3$$

LANDAU DAMPING FOR A BEAM COLLIMATED AT AN ARBITRARY NUMBER OF SIGMAS (3/17)

- **Consider the following** (nth order) distribution

$$\boxed{f\left(J_x,J_y\right) = a\left(1-\frac{J_x+J_y}{b}\right)^n}$$

$$J_x + J_y \leq J_{\max} = b$$

$$\int_{J_x=0}^{b} dJ_x \int_{J_y=0}^{b-J_x} dJ_y\, f\left(J_x,J_y\right) = \frac{a b^2}{\left(n+1\right)\left(n+2\right)} = 1$$

$$< J_x > = \int_{J_x=0}^{b} J_x\, dJ_x \int_{J_y=0}^{b-J_x} dJ_y\, f\left(J_x,J_y\right) = \frac{a b^3}{\left(n+1\right)\left(n+2\right)\left(n+3\right)} = \varepsilon$$

=> $\boxed{b = \left(n+3\right)\varepsilon}$ $\boxed{a = \frac{\left(n+1\right)\left(n+2\right)}{b^2}}$

LANDAU DAMPING FOR A BEAM COLLIMATED AT AN ARBITRARY NUMBER OF SIGMAS (4/17)

- **The horizontal profile is given by**

$$g(x) = \frac{1}{2\pi} \int_{-\sqrt{2b-x^2}}^{\sqrt{2b-x^2}} dp_x \, f(J_x)$$

$$J_x = \frac{x^2 + p_x^2}{2}$$

$$f(J_x) = \int_0^{b-J_x} dJ_y \, f(J_x, J_y) = \frac{a\,b}{n+1} \left(1 - \frac{J_x}{b}\right)^{n+1}$$

=>

$$g(x) = \frac{ab}{2\pi(n+1)} \int_{-\sqrt{2b-x^2}}^{\sqrt{2b-x^2}} \left(1 - \frac{x^2 + p_x^2}{2b}\right)^{n+1} dp_x$$

$$= \frac{2(n+2)\left[2^{n+1}(n+1)!\right]^2}{\pi\sqrt{2b}\,(2n+3)!} \left(1 - \frac{x^2}{2b}\right)^{n+\frac{3}{2}}$$

LANDAU DAMPING FOR A BEAM COLLIMATED AT AN ARBITRARY NUMBER OF SIGMAS (5/17)

Therefore, the profile extends up to $\boxed{\sqrt{2b}}$

=> In the case of a beam profile extending up to 6 σ (it is the settings for the LHC collimators), one should have

$$\sqrt{2b} = 6\,\sigma \quad \Rightarrow \quad n = 15$$

- If n = 2, it means that horizontal profile extends up to 3.2 σ
- In the case of a Gaussian distribution, one has

$$f\left(J_x, J_y\right) = \frac{1}{\varepsilon^2}\, e^{-\frac{\left(J_x + J_y\right)}{\varepsilon}} \qquad g\left(x\right) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{x^2}{2\sigma^2}}$$

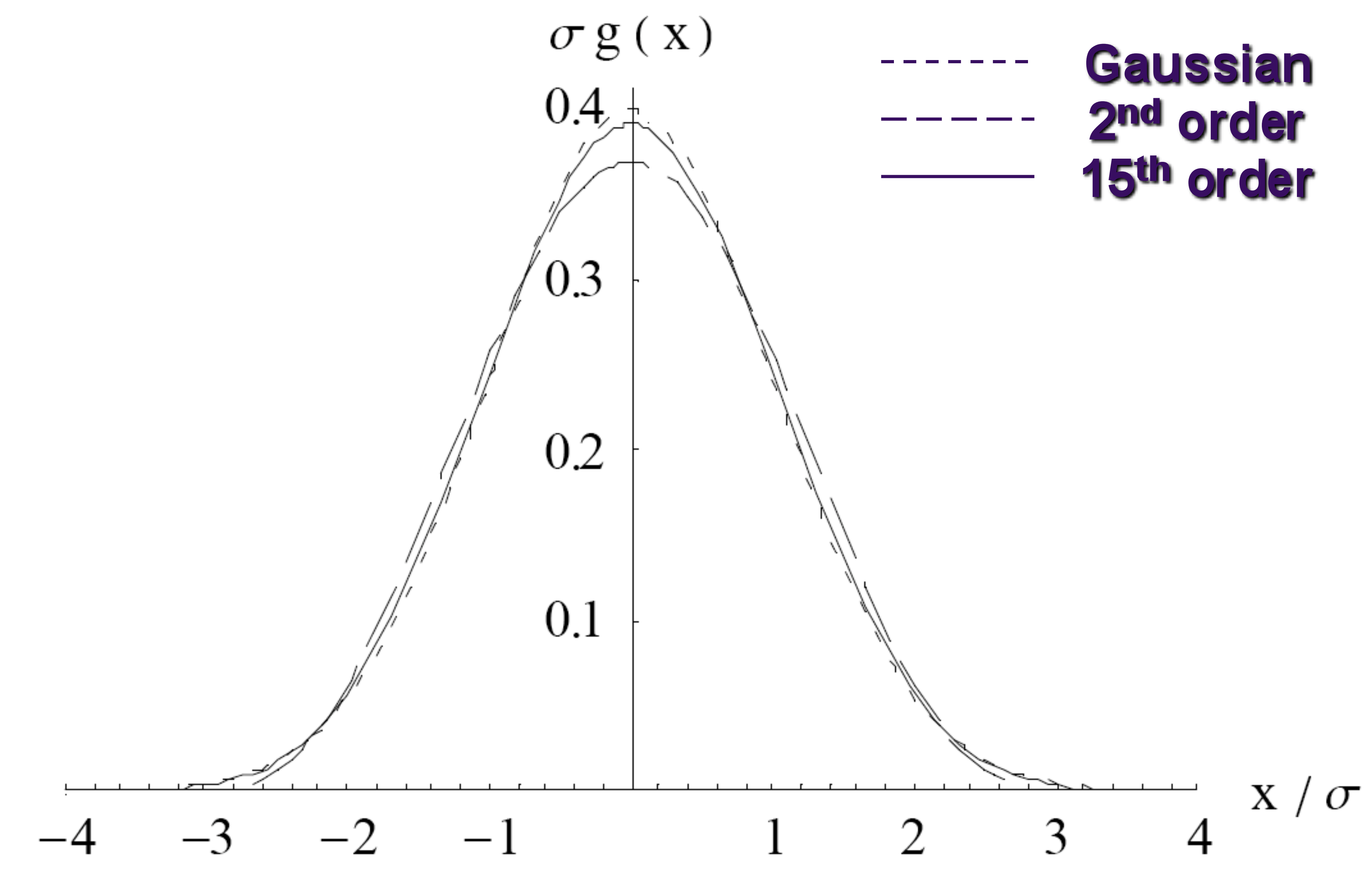
LANDAU DAMPING FOR A BEAM COLLIMATED AT AN ARBITRARY NUMBER OF SIGMAS (6/17)

- **Note that the nth order distribution function tends to the Gaussian distribution function, when n tends to infinity. This can be easily found by taking the logarithm and expanding it (** $\boxed{\operatorname{Log}(1-x) \approx -x}$ **)**

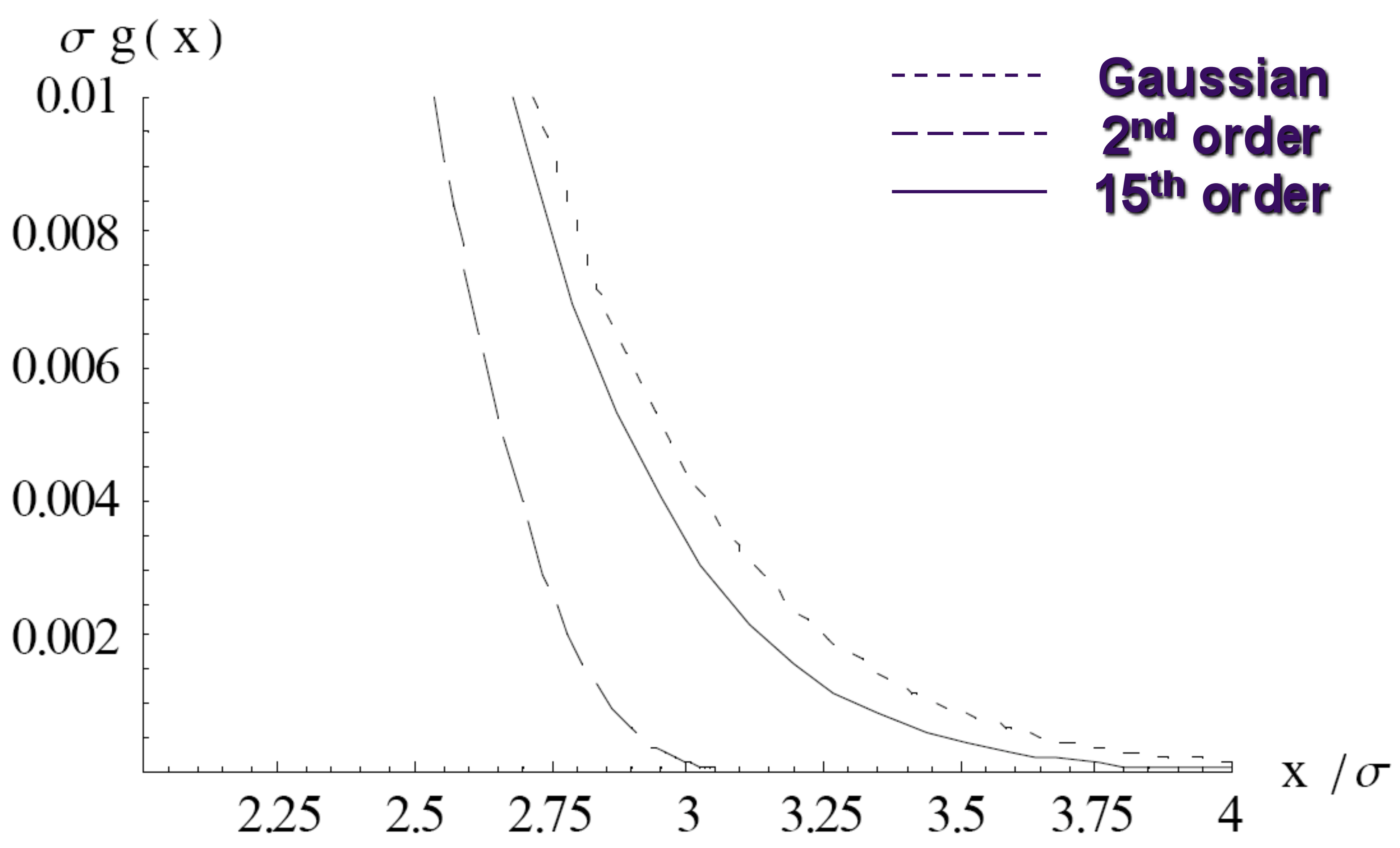
$$f\left(J_x, J_y\right) = \frac{(n+1)(n+2)}{(n+3)^2\,\varepsilon^2}\left(1-\frac{J_x+J_y}{(n+3)\,\varepsilon}\right)^n$$

$$\log\left[f\left(J_x, J_y\right)\right] = \underbrace{\log\left[\frac{(n+1)(n+2)}{(n+3)^2}\right]}_{\to\, 0 \text{ when } n \to \infty} - \log\left(\varepsilon^2\right) + \underbrace{n \log\left(1-\frac{J_x+J_y}{(n+3)\,\varepsilon}\right)}_{\to\, -\frac{J_x+J_y}{\varepsilon} \text{ when } n \to \infty}$$

LANDAU DAMPING FOR A BEAM COLLIMATED AT AN ARBITRARY NUMBER OF SIGMAS (7/17)



LANDAU DAMPING FOR A BEAM COLLIMATED AT AN ARBITRARY NUMBER OF SIGMAS (8/17)



LANDAU DAMPING FOR A BEAM COLLIMATED AT AN ARBITRARY NUMBER OF SIGMAS (9/17)

- **Stability diagram**

$$\Delta Q^{x}_{coh} = -\frac{a_0}{n\,a\,b}\, I_n^{-1}(c,q)$$

$$I_n(c,q) = \int_{J_x=0}^{1} dJ_x \int_{J_y=0}^{1-J_x} dJ_y \, \frac{J_x\left(1-J_x-J_y\right)^{n-1}}{q+J_x+c\,J_y}$$

$$q = \frac{Q_c - Q_0 - m\,Q_s}{-b\,a_0}$$

***m* = 0 for coasting beams**

$$c = \frac{b_0}{a_0}$$

- **For n = 2, this can be solved analytically and one obtains**

$$I_2(c,q) = -\left\{ \begin{array}{l} (c+q)^3 \,\mathrm{Log}[1+q] - (c+q)^3 \,\mathrm{Log}[c+q] + (c-1) \\ \left\{ \begin{array}{l} c\left[c + 2cq + (2c-1)\,q^2\right] \\ +(c-1)\,q^2\,(3c+q+2cq)\left(\mathrm{Log}[q] - \mathrm{Log}[1+q]\right) \end{array} \right\} \end{array} \right\} \Big/ \left[6\,(c-1)^2\, c^2\right]$$

LANDAU DAMPING FOR A BEAM COLLIMATED AT AN ARBITRARY NUMBER OF SIGMAS (10/17)

- For the Gaussian distribution, also

$$\Delta Q_{coh}^{x} = - a_0 \, \varepsilon \, K^{-1}(c,p)$$

$$K(c,p) = \int_{J_x=0}^{\infty} dJ_x \int_{J_y=0}^{\infty} dJ_y \, \frac{J_x \, e^{-(J_x + J_y)}}{p + J_x + c \, J_y}$$

$$p = \frac{Q_c - Q_0 - m \, Q_s}{- a_0 \, \varepsilon}$$

$$E_1(z) = \int_{t=z}^{t=\infty} \frac{e^{-t}}{t} dt$$

m = 0 for coasting beams

$$K(c,p) = \frac{1 - c - (p + c - c\,p) \, e^{p} \, E_1(p) + c \, e^{p/c} \, E_1(p/c)}{(1-c)^2}$$

- For the 15th order distribution, it was solved numerically

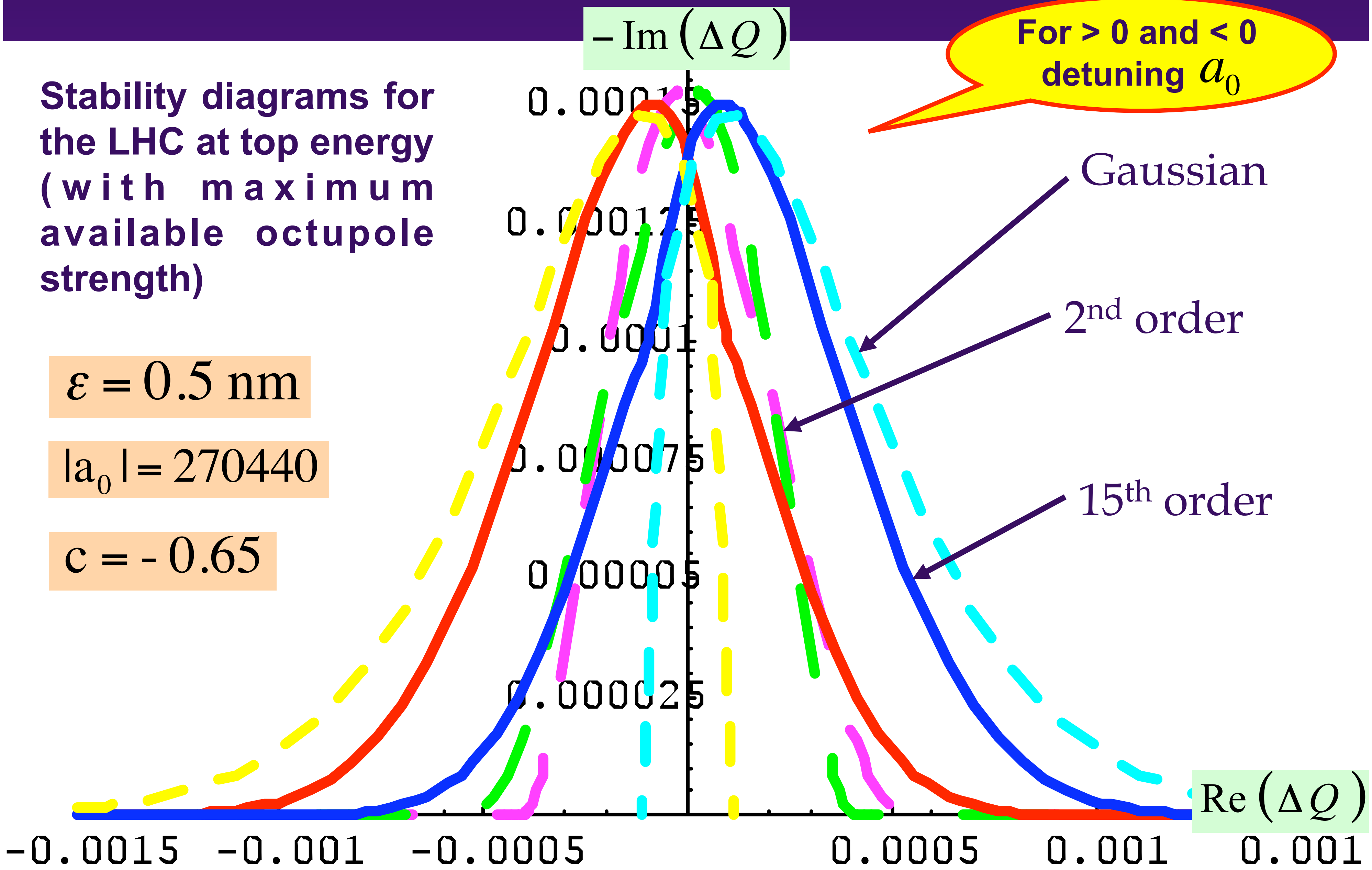
LANDAU DAMPING FOR A BEAM COLLIMATED AT AN ARBITRARY NUMBER OF SIGMAS (11/17)

Stability diagrams for the LHC at top energy (with maximum available octupole strength)

$\varepsilon = 0.5$ nm

$|a_0| = 270440$

$c = -0.65$



LANDAU DAMPING FOR A BEAM COLLIMATED AT AN ARBITRARY NUMBER OF SIGMAS (12/17)

- **Distribution with more populated tails than the Gaussian (this case may apply in reality in proton machines, where several diffusive mechanisms can take place)**

$$\boxed{f\left(J_x, J_y\right) = a\left(1 - \frac{J_x + J_y}{b}\right)^n + d\left(1 - \frac{J_x + J_y}{b}\right)^p}$$

=> Here we impose that the beam profile extends up to 6 σ (it is the settings for the LHC collimators)

$$\int_{J_x=0}^{b} dJ_x \int_{J_y=0}^{b-J_x} dJ_y\, f\left(J_x, J_y\right) = \frac{a\, b^2}{(n+1)(n+2)} + \frac{d\, b^2}{(p+1)(p+2)} = 1$$

$$\int_{J_x=0}^{b} J_x\, dJ_x \int_{J_y=0}^{b-J_x} dJ_y\, f\left(J_x, J_y\right) = \frac{a\, b^3}{(n+1)(n+2)(n+3)} + \frac{d\, b^3}{(p+1)(p+2)(p+3)} = \varepsilon$$

LANDAU DAMPING FOR A BEAM COLLIMATED AT AN ARBITRARY NUMBER OF SIGMAS (13/17)

=>

$$a = \frac{(n+1)(n+2)(n+3)(15-p)}{18\, b^2\, (n-p)}$$

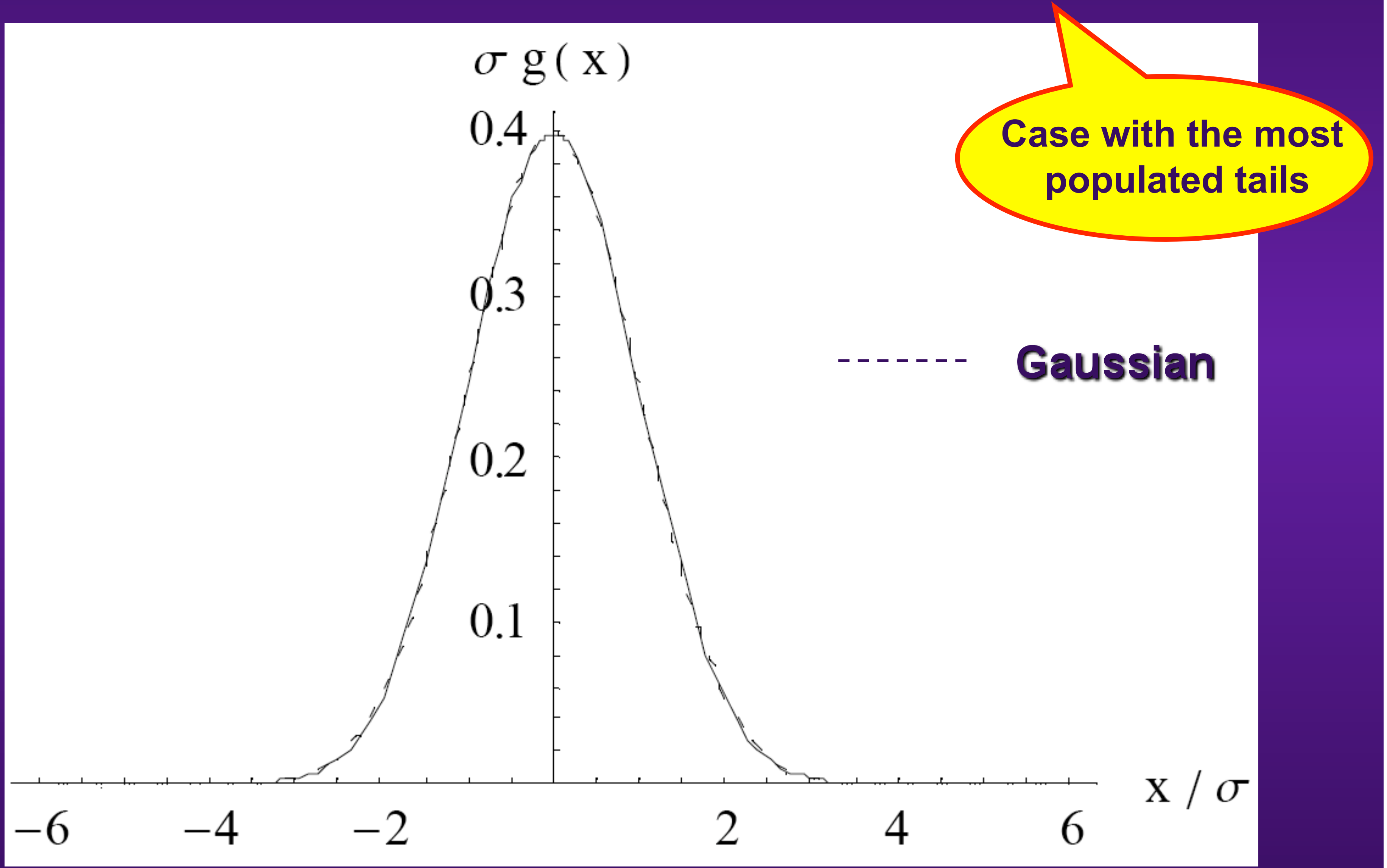
$$d = \frac{(p+1)(p+2)(p+3)(n-15)}{18\, b^2\, (n-p)}$$

n must be > 15
and p < 15 (to guarantee a positive density)

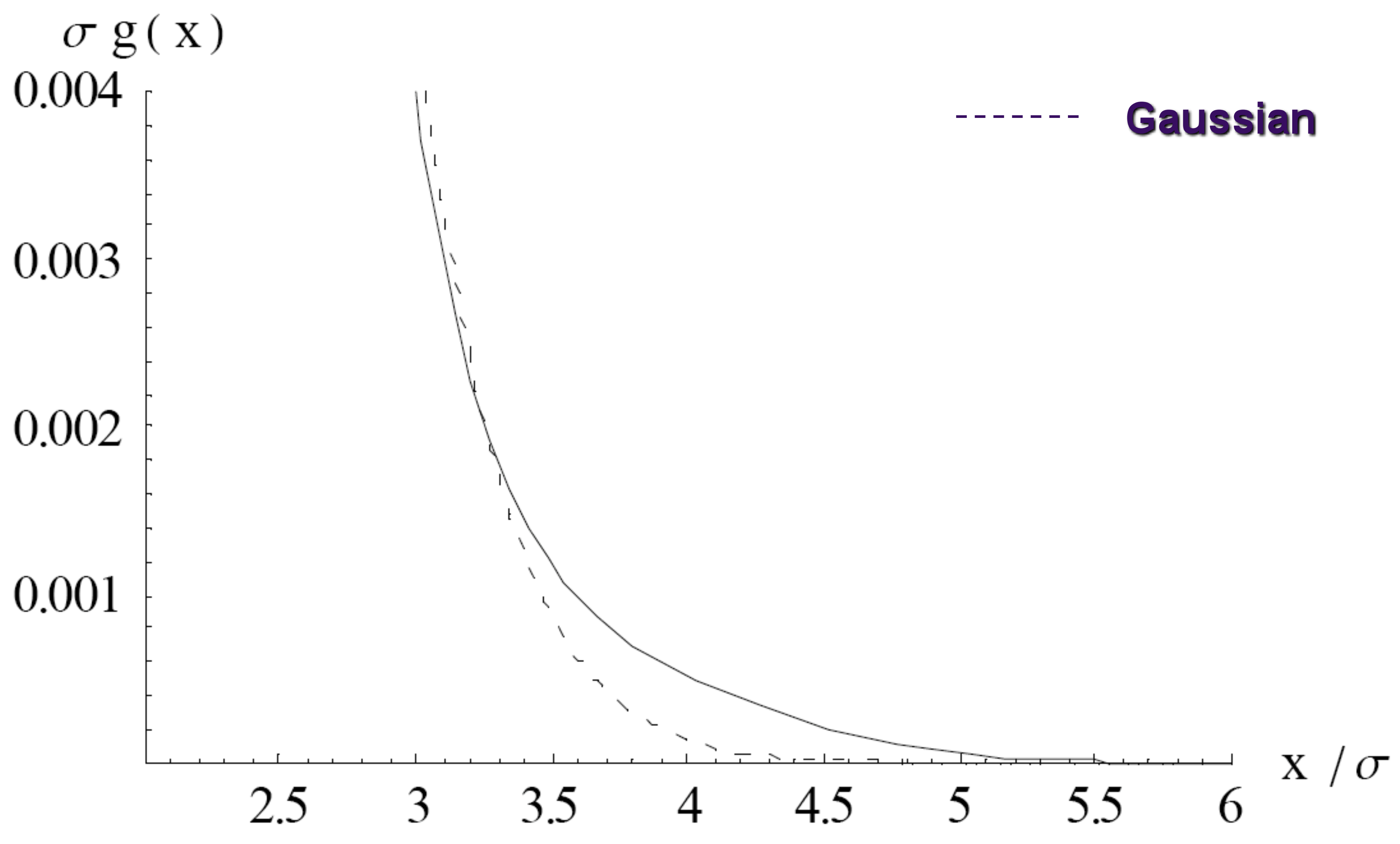
$$g(x) = \frac{1}{9\pi (n-p)\sqrt{2b}} \left\{ \frac{(n+2)(n+3)(15-p)\left[2^{n+1}(n+1)!\right]^2}{(2n+3)!} \left(1-\frac{x^2}{2b}\right)^{n+\frac{3}{2}} + \frac{(p+2)(p+3)(n-15)\left[2^{p+1}(p+1)!\right]^2}{(2p+3)!} \left(1-\frac{x^2}{2b}\right)^{p+\frac{3}{2}} \right\}$$

LANDAU DAMPING FOR A BEAM COLLIMATED AT AN ARBITRARY NUMBER OF SIGMAS (14/17)

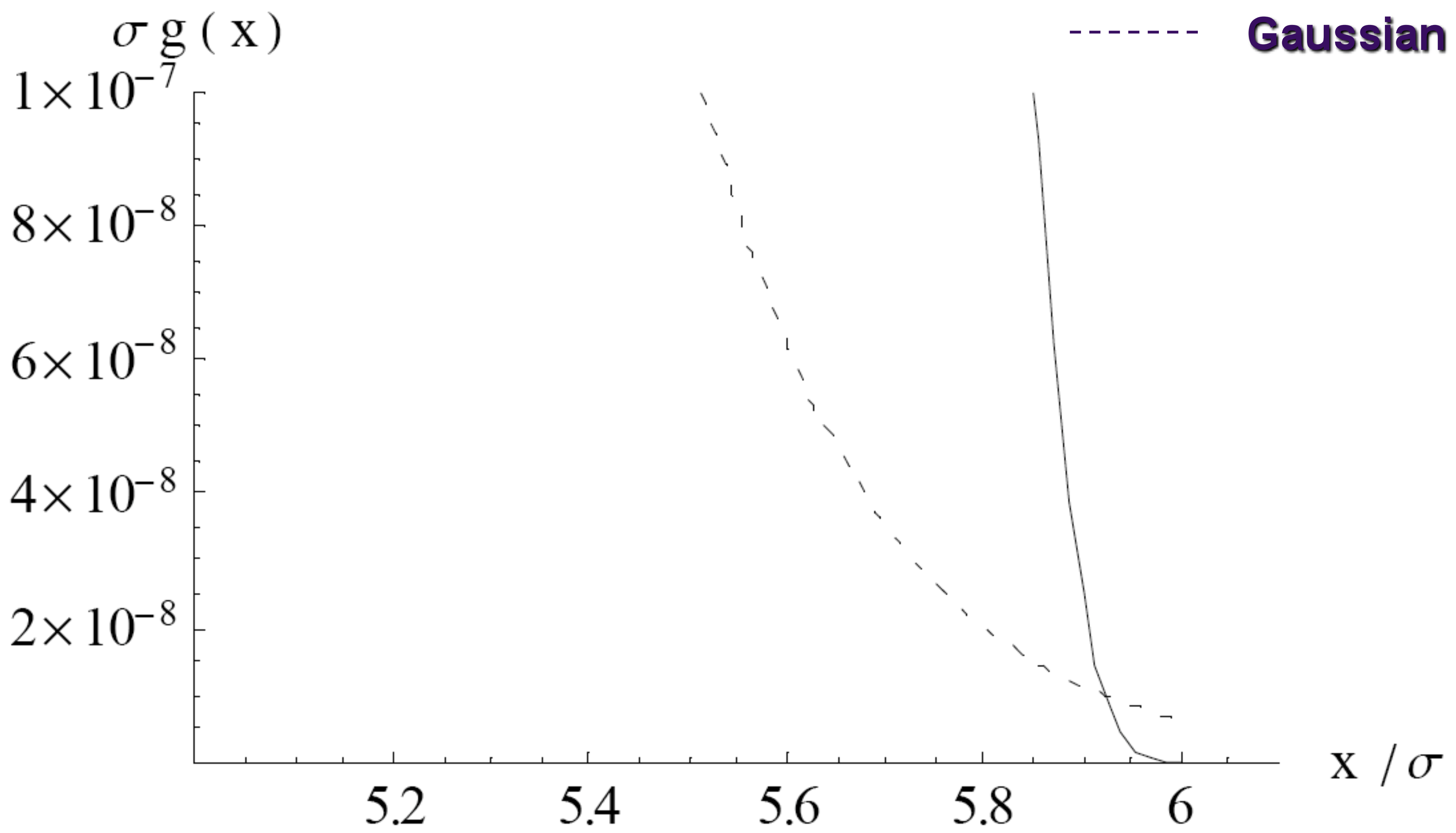
=> Let's consider as an example the case n = 16 and p = 2



LANDAU DAMPING FOR A BEAM COLLIMATED AT AN ARBITRARY NUMBER OF SIGMAS (15/17)



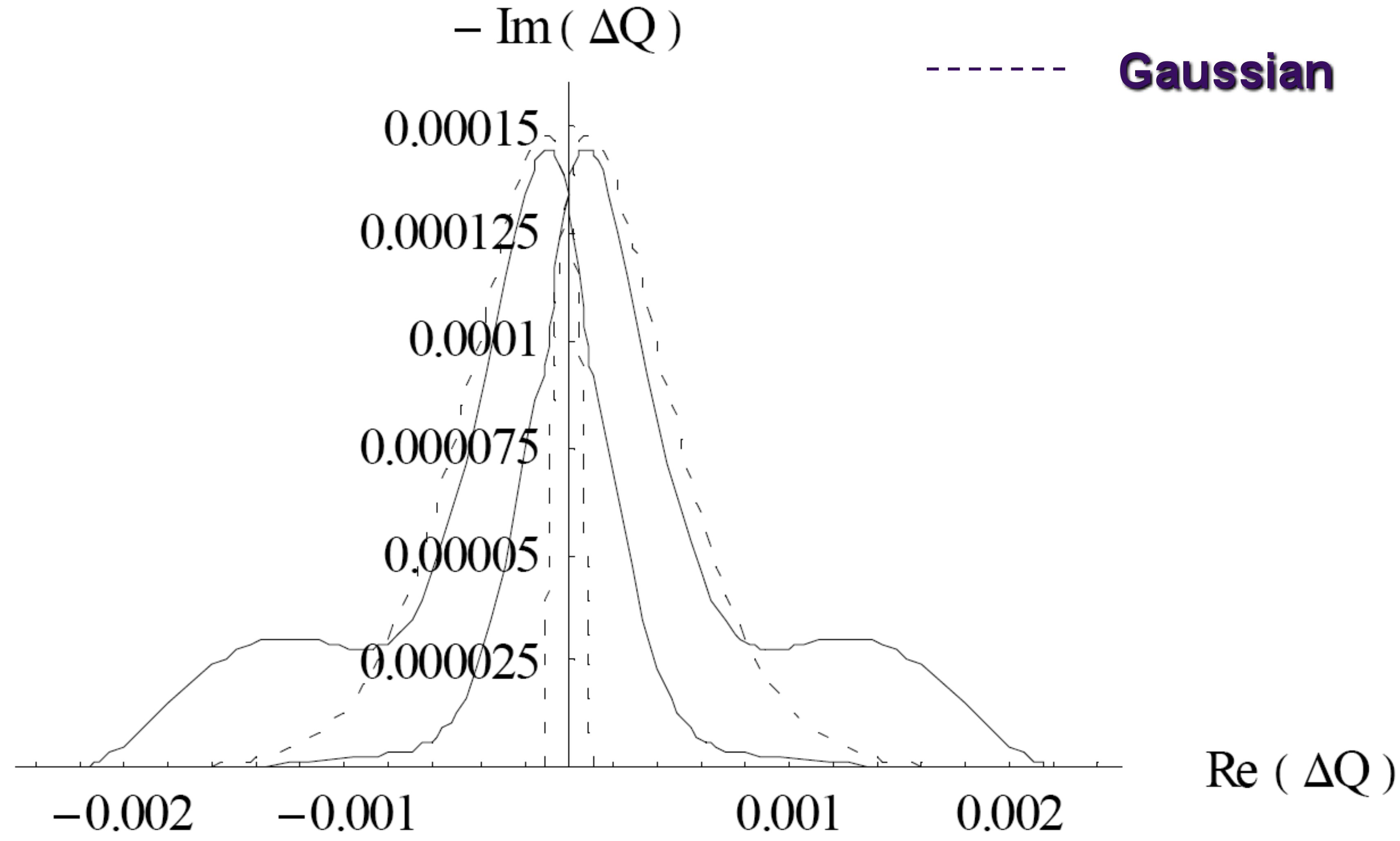
LANDAU DAMPING FOR A BEAM COLLIMATED AT AN ARBITRARY NUMBER OF SIGMAS (16/17)



LANDAU DAMPING FOR A BEAM COLLIMATED AT AN ARBITRARY NUMBER OF SIGMAS (17/17)

- **The dispersion relation writes**

$$\Delta Q_{coh}^{x} = - \frac{a_0 / b}{n\, a\, I_n(c,q) + p\, d\, I_p(c,q)}$$



LANDAU DAMPING WITH 2-DIMENSIONAL BETATRON TUNE SPREAD FROM BOTH OCTUPOLES AND NONLINEAR SPACE CHARGE (1/7)

- **If the nonlinear part of space charge is also taken into account the situation is even more involved and the dispersion relation is given by (neglecting the momentum spread but treating "correctly" the two-dimensional betatron tune spread introduced by Landau octupoles)**

$$1 = - \int_{J_x = 0}^{+\infty} dJ_x \int_{J_y = 0}^{+\infty} dJ_y \, \frac{J_x \, \dfrac{\partial f\left(J_x, J_y\right)}{\partial J_x} \left[\boxed{\Delta Q_{coh}^{x} - \Delta Q_{incoh}^{x}\left(J_x, J_y\right)} \right]}{Q_c - Q_x\left(J_x, J_y\right) - m\, Q_s}$$

***m* = 0 for coasting beams**

$$f\left(J_x, J_y\right) = \frac{12}{J_{\max}^2} \left(1 - \frac{J_x + J_y}{J_{\max}} \right)^2$$

$$J_x + J_y \leq J_{\max} = 5\,\sigma^2$$

LANDAU DAMPING WITH 2-DIMENSIONAL BETATRON TUNE SPREAD FROM BOTH OCTUPOLES AND NONLINEAR SPACE CHARGE (2/7)

$$\boxed{Q_x\left(J_x, J_y\right) = Q_{x0}\left(J_x, J_y\right) + \Delta Q_{incoh}^{x}\left(J_x, J_y\right)}$$

$$Q_{x0}\left(J_x, J_y\right) = Q_{x00} + a\, J_x + b\, J_y$$

$$a = \frac{3}{8\,\pi} \int \beta_x^2 \frac{O_3}{B\rho}\, ds \qquad b = -\frac{3}{8\,\pi} \int 2\,\beta_x\,\beta_y \frac{O_3}{B\rho}\, ds$$

- **The computation of the incoherent (nonlinear) tune shift has to be self-consistent (with the assumed distribution function) and it corresponds to the case computed in the SPACE CHARGE course**

$$\int_{p_x} dp_x \int_{p_y} dp_y\, f\left(J_x, J_y\right) = \frac{8\,\pi}{5\,\sigma^2}\left(1 - \frac{x^2 + y^2}{10\,\sigma^2}\right)^3$$

LANDAU DAMPING WITH 2-DIMENSIONAL BETATRON TUNE SPREAD FROM BOTH OCTUPOLES AND NONLINEAR SPACE CHARGE (3/7)

=> Reminder:

$$\Delta Q_{incoh}^{x}\left(j_x, j_y\right) = \Delta_0 \left[1 - \frac{9}{8} j_x - \frac{3}{4} j_y + \frac{5}{8} j_x^2 + \frac{3}{4} j_x \, j_y + \frac{3}{8} j_y^2 - \frac{35}{256} j_x^3 - \frac{15}{64} j_x^2 \, j_y - \frac{27}{128} j_x \, j_y^2 - \frac{5}{64} j_y^3 \right]$$

$$j_x = J_x / J_{\max} \qquad j_y = J_y / J_{\max}$$

with

$$\Delta_0 = - \frac{N_b \, r_p}{5 \, \pi \, B \, \beta \, \gamma^2 \varepsilon_{rms}^{norm}}$$

$$J_{\max} = 5 \, \sigma^2$$

LANDAU DAMPING WITH 2-DIMENSIONAL BETATRON TUNE SPREAD FROM BOTH OCTUPOLES AND NONLINEAR SPACE CHARGE (4/7)

- **Knowing the expression of the non-linear space-charge tune shift, the (exact) dispersion relation can be obtained**
- **A reasonable approximation of the self-consistent non-linear space-charge tune shift is given by taking into account only the linear terms in the betatron action variables (adapting the coefficients!) => The dispersion relation is solved in this case**

$$\Delta Q^x_{incoh}\left(J_x, J_y\right) = \Delta_0 + \Delta_a \, J_x + \Delta_b \, J_y$$

$$\Delta Q^x_{coh} = \Delta_0 + \frac{1}{K_1\left(c_1, q\right)} \times \left[\frac{S_1}{24} + J_{\max} \, \Delta_a \, K_2\left(c_1, q\right) + J_{\max} \, \Delta_b \, K_3\left(c_1, q\right) \right]$$

$$a_1 = a + \Delta_a \qquad b_1 = b + \Delta_b \qquad c_1 = \frac{b_1}{a_1} \qquad S_1 = -\, a_1 \, J_{\max}$$

LANDAU DAMPING WITH 2-DIMENSIONAL BETATRON TUNE SPREAD FROM BOTH OCTUPOLES AND NONLINEAR SPACE CHARGE (5/7)

***m* = 0 for coasting beams**

$$q = \frac{Q_c - \left(Q_{x00} + m Q_s + \Delta_0 \right)}{S_1}$$

$$K_1(c_1, q) = -\frac{1}{6\, c_1^2 \,(c_1 - 1)^2} \times \begin{Bmatrix} (c_1 + q)^3 \log(1+q) - (c_1+q)^3 \log(c_1 + q) \\ + (c_1 - 1) \left\{ c_1 \left[c_1 + 2 c_1 q + (2 c_1 - 1)\, q^2 \right] \right\} \\ + (c_1 - 1)\, q^2 \,(3 c_1 + q + 2\, c_1 q) \left[\log(q) - \log(1+q) \right] \end{Bmatrix}$$

$$K_2(c_1, q) = -\frac{1}{24\, c_1^2 \,(c_1 - 1)^3} \times \begin{Bmatrix} (c_1 - 1)\, c_1 \begin{Bmatrix} c_1 - 3 c_1^2 - 2 c_1 (c_1 + 2)\, q + c_1 (-11 + 5 c_1)\, q^2 \\ + 2 \left[1 + c_1 (-5 + 3 c_1) \right] q^3 \end{Bmatrix} \\ -2\, (c_1 + q)^4 \log(1+q) \\ +2 \begin{Bmatrix} (c_1 + q)^4 \log(c_1 + q) \\ + (-1 + c_1)^3 q^3 (4\, c_1 + q + 3\, c_1 q) \left[\log(q) - \log(1+q) \right] \end{Bmatrix} \end{Bmatrix}$$

$$K_3(c_1, q) = -\frac{1}{24\, c_1^3 \,(c_1 - 1)^3} \times \begin{Bmatrix} (-1 + c_1)\, c_1 \left\{ c_1^2 (1 + c_1) + 6\, c_1^2\, q + 3\, c_1 (1 + c_1)\, q^2 + 2 \left[1 + (-1 + c_1)\, c_1 \right] q^3 \right\} \\ +2\, (c_1 + q)^3 (c_1 - q + 2\, c_1 q) \log(1+q) - 2\, (c_1 + q)^3 (c_1 - q + 2\, c_1 q) \log(c_1 + q) \\ +2\, (-1 + c_1)^3\, q^3 \left[q + c_1 (2 + q) \right] \left[\log(q) - \log(1+q) \right] \end{Bmatrix}$$

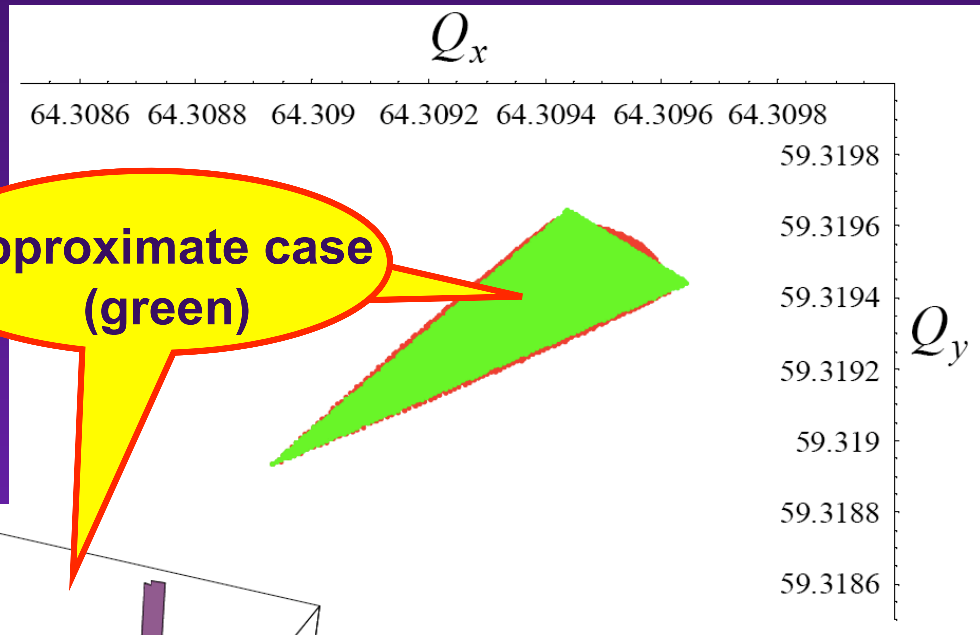
LANDAU DAMPING WITH 2-DIMENSIONAL BETATRON TUNE SPREAD FROM BOTH OCTUPOLES AND NONLINEAR SPACE CHARGE (6/7)

- **Application to the case of the LHC at injection**

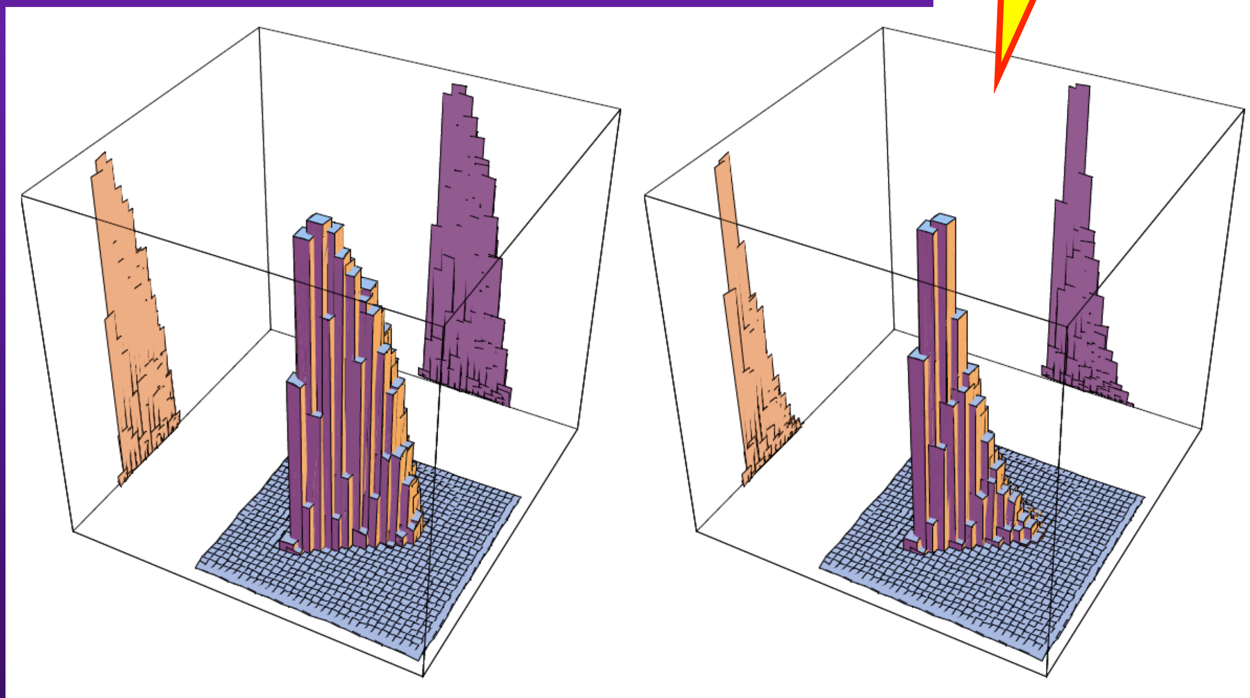
$\varepsilon = 7.8 \text{ nm}$ $\quad Q_{x00} = 64.31$

$b = \mp 4647$ $\quad Q_{y00} = 59.32$

$a = \pm \Delta Q^{x,rms}_{oct,spread} / \varepsilon = \pm 7164.2$



Approximate case (green)



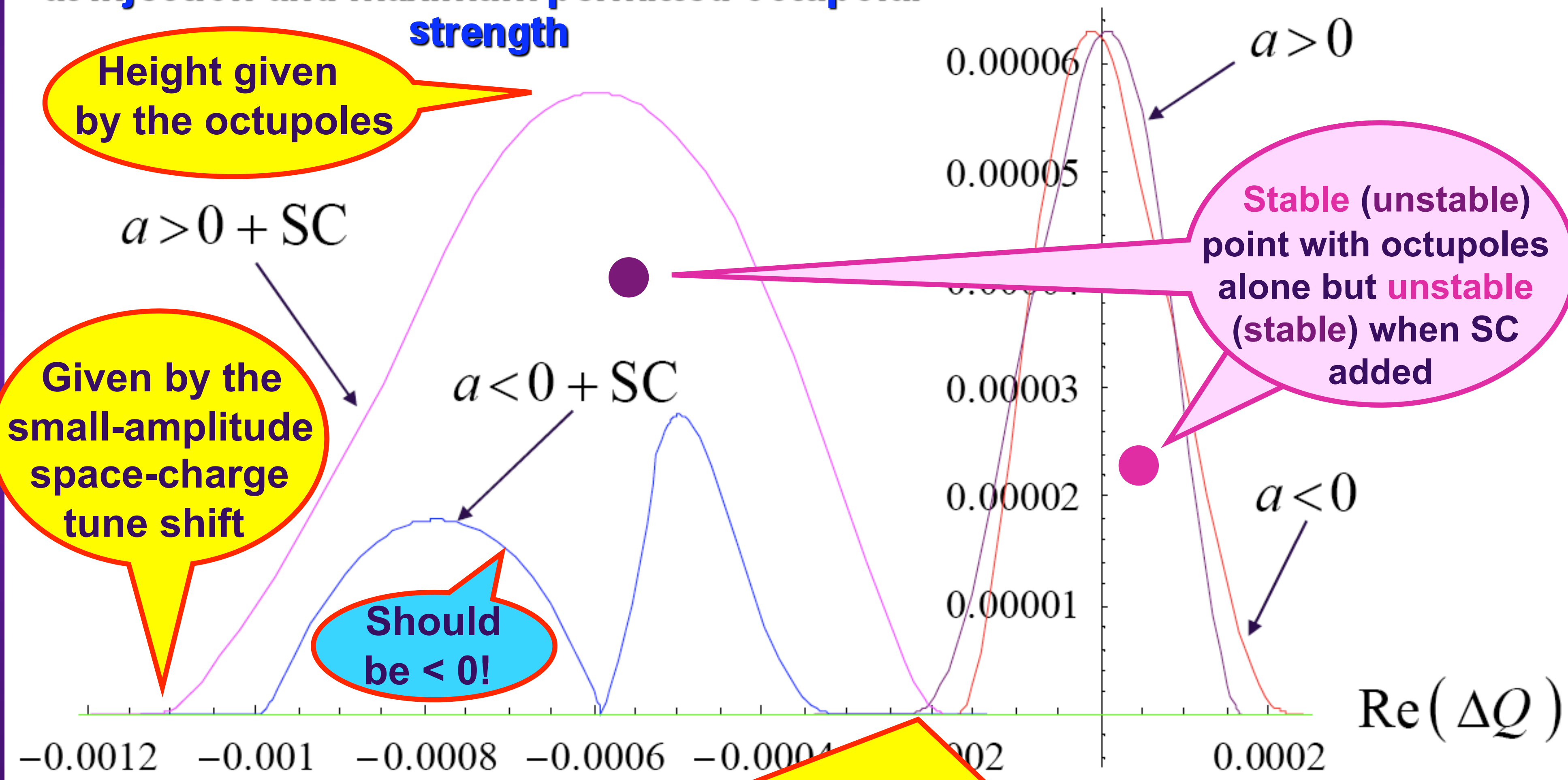
$$\Delta_0 \approx -1.1 \times 10^{-3}$$

$$\Delta_a = 18127$$

$$\Delta_b = 12948$$

LANDAU DAMPING WITH 2-DIMENSIONAL BETATRON TUNE SPREAD FROM BOTH OCTUPOLES AND NONLINEAR SPACE CHARGE (7/7)

Stability diagrams for nominal LHC parameters at injection and maximum permitted octupolar strength



Given by the large-amplitude space-charge tune shift + octupoles
=> Will move to the right with longitudinal motion

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (1/35)

- **Coupled equations of motion with wake-fields**

$$\ddot{x}_i + \Omega_i^2 \left(Q_{0x,i}^2 + 2\, Q_{0x,i}\, \Delta Q_{inc,x} \right) x_i = -2\, \Omega_i^2\, Q_{0x,i} \left(\Delta Q_{coh,x} - \Delta Q_{inc,x} \right) \overline{x} + \underline{K}_i\, R_i^2\, \Omega_i^2\, y_i$$

$$\ddot{y}_i + \Omega_i^2 \left(Q_{0y,i}^2 + 2\, Q_{0y,i}\, \Delta Q_{inc,y} \right) y_i = -2\, \Omega_i^2\, Q_{0y,i} \left(\Delta Q_{coh,y} - \Delta Q_{inc,y} \right) \overline{y} + \underline{K}_i\, R_i^2\, \Omega_i^2\, x_i$$

$$\underline{K}_i = \left(e \,/\, p_i \right) \left(\partial B_{x,i} \,/\, \partial x_i \right)$$

Normalised skew gradient

- **Using the normalised (Courant-Snyder) coordinates and angle, given by**

$$\eta_i = x_i\, \beta_{0x,i}^{-1/2}(s) \qquad \zeta_i = y_i\, \beta_{0y,i}^{-1/2}(s) \qquad \phi_i = Q_{0x,i}^{-1} \int_0^s \beta_{0x,i}^{-1}(t)\, dt \approx Q_{0y,i}^{-1} \int_0^s \beta_{0y,i}^{-1}(t)\, dt$$

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (2/35)

$$\frac{d^2\eta_i}{d\phi_i^2} + Q_{x,i}^2\,\eta_i = \beta_{0x,i}^{3/2}\,\beta_{0y,i}^{1/2}\,Q_{0x,i}^2\,\underline{K}_i\,\zeta_i - \left(\Delta Q_{coh,x} - \Delta Q_{inc,x}\right)\frac{2\,Q_{0x,i}^3\,\beta_{0x,i}^2}{R_i^2}\,\overline{\eta}$$

$$\frac{d^2\zeta_i}{d\phi_i^2} + Q_{y,i}^2\,\zeta_i = \beta_{0y,i}^{3/2}\,\beta_{0x,i}^{1/2}\,Q_{0y,i}^2\,\underline{K}_i\,\eta_i - \left(\Delta Q_{coh,y} - \Delta Q_{inc,y}\right)\frac{2\,Q_{0y,i}^3\,\beta_{0y,i}^2}{R_i^2}\,\overline{\zeta}$$

$$Q_{x,i} = Q_{0x,i} + \Delta Q_{inc,x} \qquad Q_{y,i} = Q_{0y,i} + \Delta Q_{inc,y}$$

- **Assuming**

$$\beta_{0x,i} \approx R_i / Q_{0x,i} \approx R / Q_{0x0} \qquad \beta_{0y,i} \approx R_i / Q_{0y,i} \approx R / Q_{0y0} \qquad \phi_i = \Omega_i\,t = \Omega_0\,t = \phi$$

$$R_i = R \qquad \underline{K}_i = \underline{K}_0$$

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (3/35)

$$\frac{d^2\eta_i}{d\phi^2} + Q_{x,i}^2\,\eta_i = -j\,\frac{e\,\beta\,I\,Z_x}{2\,\pi\,R\,m_0\,\gamma\,\Omega_0^2}\,\overline{\eta} + R^2\left(\frac{Q_{0x0}}{Q_{0y0}}\right)^{1/2}\underline{K}_0\,\zeta_i$$

$$\frac{d^2\zeta_i}{d\phi^2} + Q_{y,i}^2\,\zeta_i = -j\,\frac{e\,\beta\,I\,Z_y}{2\,\pi\,R\,m_0\,\gamma\,\Omega_0^2}\,\overline{\zeta} + R^2\left(\frac{Q_{0y0}}{Q_{0x0}}\right)^{1/2}\underline{K}_0\,\eta_i$$

- **In the following, transverse betatron frequency spreads specified by externally given beam frequency spectra are assumed. The ensemble of particles has spectra with distribution functions which are supposed to be uncorrelated and normalised to unity**

$$\int_{-\infty}^{+\infty}\rho_x\left(\omega_{x,i}\right)\,d\omega_{x,i} = 1 \qquad \int_{-\infty}^{+\infty}\rho_y\left(\omega_{y,i}\right)\,d\omega_{y,i} = 1$$

- **Moreover, in a circular machine, linear coupling is periodic in ϕ with period 2π, and thus can be expanded into Fourier series**

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (4/35)

$$\underline{K}_0(\phi) = \sum_{l=-\infty}^{l=+\infty} \hat{\underline{K}}_0(l)\, e^{j l \phi}$$

$$\hat{\underline{K}}_0(l) = \frac{1}{2\pi} \int_0^{2\pi} \underline{K}_0(\phi)\, e^{-j l \phi}\, d\phi$$

- **Considering only the dominant Fourier component of the coupling (*l*) and following the standard procedure of identifying normal mode frequencies, yields particular solutions of the form**

$$\eta_i = \mathrm{H}_i\, e^{j Q_c \phi}$$

$$\zeta_i = \mathrm{Z}_i\, e^{j (Q_c - l) \phi}$$

$$\mathrm{H}_i = \frac{1}{Q_{x,i}^2 - Q_c^2} \left[R^2 \left(\frac{Q_{0x0}}{Q_{0y0}} \right)^{1/2} \hat{\underline{K}}_0(l)\, \mathrm{Z}_i + \frac{2\omega_{x0}}{\Omega_0^2} \left(-U_x + j V_x \right) \overline{\mathrm{H}} \right]$$

$$\mathrm{Z}_i = \frac{1}{Q_{y,i}^2 - (Q_c - l)^2} \left[R^2 \left(\frac{Q_{0y0}}{Q_{0x0}} \right)^{1/2} \hat{\underline{K}}_0(-l)\, \mathrm{H}_i + \frac{2\omega_{y0}}{\Omega_0^2} \left(-U_y + j V_y \right) \overline{\mathrm{Z}} \right]$$

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (5/35)

- **After integrating over both spectra and equating the terms $\overline{\mathrm{H}} / \overline{\mathrm{Z}}$ in both equations, yields**

$$\left[\left(\int_{-\infty}^{+\infty} \frac{\rho_x\left(\omega_{x,i}\right) d\omega_{x,i}}{\omega_c - \omega_{x,i}}\right)^{-1} - U_x + j V_x\right] \times$$

$$\left[\left(\int_{-\infty}^{+\infty} \frac{\rho_y\left(\omega_{y,i}\right) d\omega_{y,i}}{\omega_c - l\Omega_0 - \omega_{y,i}}\right)^{-1} - U_y + j V_y\right] = \frac{\left|\underline{\hat{K}}_0(l)\right|^2 R^4 \, \Omega_0^4}{4 \, \omega_{x0} \, \omega_{y0}}$$

$$\omega_c = \Omega_0 \, Q_c$$

- **The wake field terms must be evaluated at the local collective frequencies, given approximately by**

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (6/35)

$$\omega_1 \approx \left(n_x + Q_{x0} \right) \Omega_0 \qquad \omega_2 \approx \left(n_y + Q_{y0} \right) \Omega_0$$

where the transverse azimuthal mode numbers are related by

$$n_x = n_y - l$$

- Solution of the 2-dimensional dispersion relation considering Lorentzian spectra (already discussed before)

=>

$$\left[\omega_c - \omega_{x0} - U_x - j \left(\delta\omega_x - V_x \right) \right] \times \left[\omega_c - \omega_{y0} - l \Omega_0 - U_y - j \left(\delta\omega_y - V_y \right) \right] = \frac{\left| \underline{\hat{K}}_0 \left(l \right) \right|^2 R^4 \, \Omega_0^4}{4 \, \omega_{x0} \, \omega_{y0}}$$

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (7/35)

- **The imaginary parts of the two coherent oscillation frequencies are given by**

$$\mathrm{Im}\left(\omega_{c1,2}\right) = \left(\delta\omega_{x,y} - V_{x,y}\right) \pm \frac{\left(\delta\omega_y - V_y - \delta\omega_x + V_x\right)}{2}\, C\left(a,\delta\right)$$

Coupling (or sharing) function

with

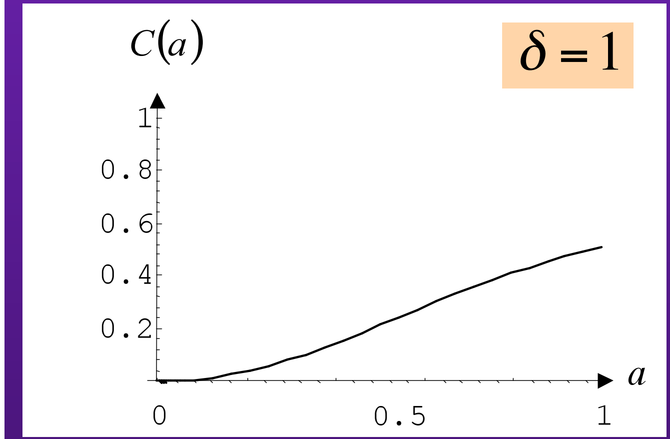
$$C\left(a,\delta\right) = 1 - \frac{1}{\sqrt{2}}\sqrt{1 - 4\,a^2 - \delta^2 + \sqrt{\left(-1 + 4\,a^2 + \delta^2\right)^2 + 4\,\delta^2}}$$

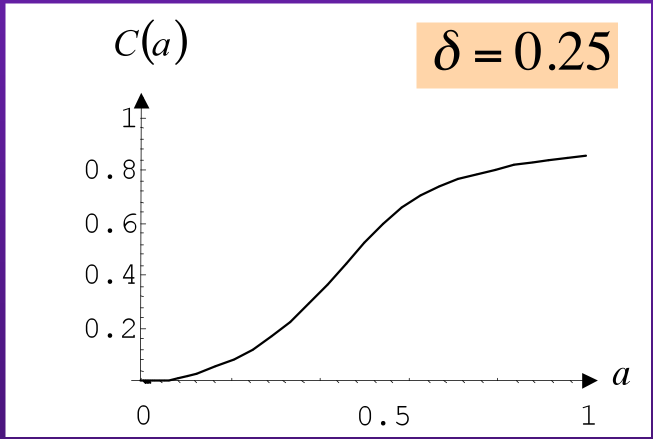
$$a = \frac{\left|\hat{\underline{K}}_0\left(l\right)\right| R^2\,\Omega_0^2}{2\sqrt{\omega_{x0}\,\omega_{y0}}\left|\delta\omega_y - V_y - \delta\omega_x + V_x\right|}$$

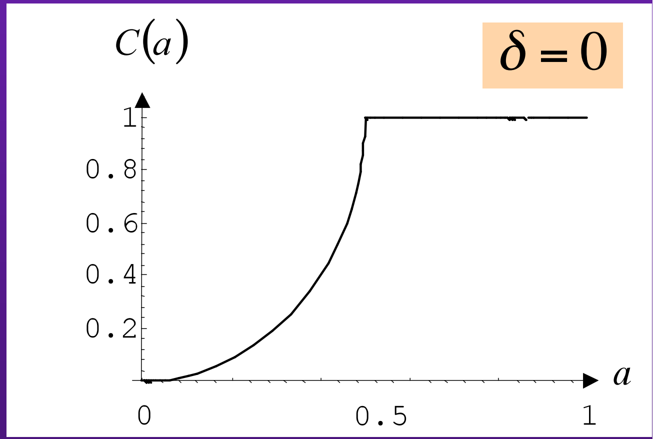
$$\delta = \frac{\Omega_0\left|Q_h - Q_v - l\right|}{\left|\delta\omega_y - V_y - \delta\omega_x + V_x\right|}$$

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (8/35)

where $Q_{h,v} = \left(\omega_{x0,y0} + U_{x,y} \right) / \Omega_0$ **are the horizontal and vertical coherent tunes in the presence of wake fields, but in the absence of coupling**







EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (9/35)

- **Transverse stability criteria**

  - **For** $\boxed{C = 0}$ **(no coupling)**

$$\boxed{\delta\omega_x \geq V_x} \qquad \boxed{\delta\omega_y \geq V_y}$$

  - **For** $\boxed{C = 1}$ **(full coupling)**

Transfer of instability growth rates in the absence of betatron frequency spread

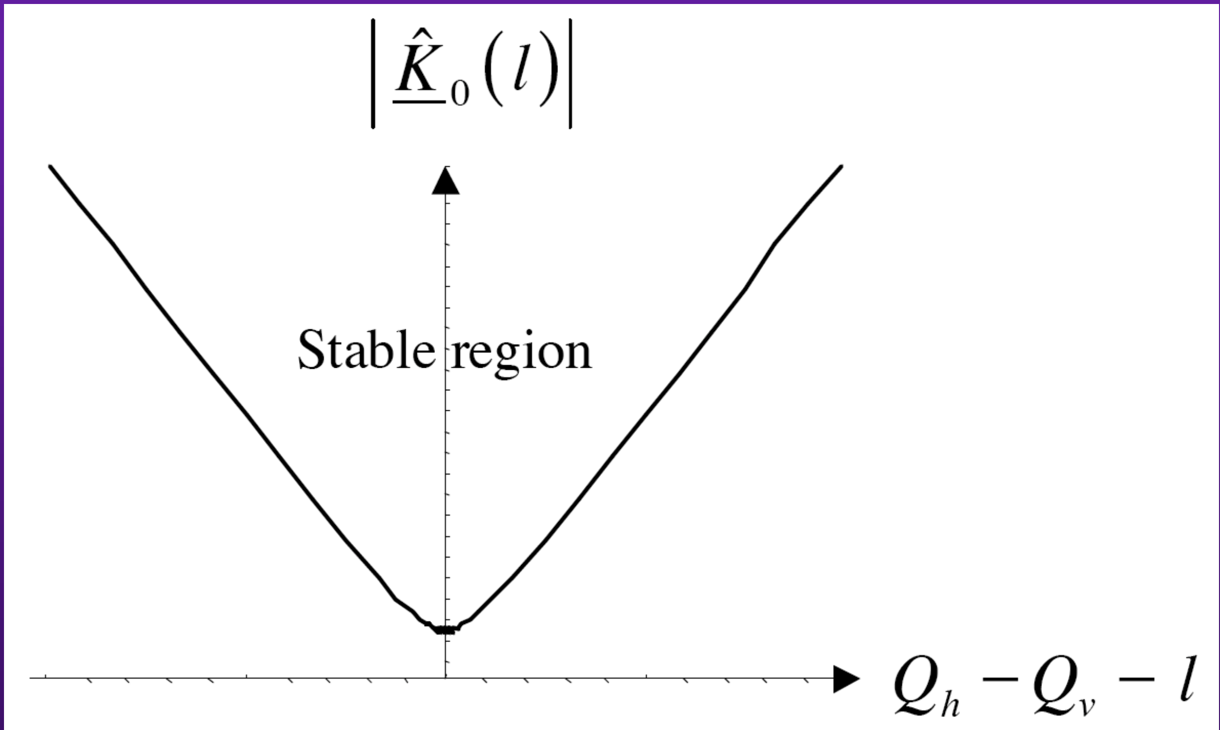
Transfer of Landau damping (but depends on the distribution)

$$\boxed{\delta\omega_x + \delta\omega_y \geq V_x + V_y}$$

  - **Consider the interesting case of one unstable transverse plane in the absence of coupling. If the necessary condition of the previous equation is fulfilled, then it is possible to stabilize the beam in the two planes. The stabilizing values of the modulus of the Fourier coefficient of the skew gradient are given by**

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (10/35)

$$\left| \underline{\hat{K}}_0(l) \right| \geq \frac{2\left[-Q_{x0}\, Q_{y0} \left(\delta\omega_x - V_x \right) \left(\delta\omega_y - V_y \right) \right]^{1/2}}{R^2\, \Omega_0} \times \frac{\left[\left(\delta\omega_x + \delta\omega_y - V_x - V_y \right)^2 + \Omega_0^2 \left(Q_h - Q_v - l \right)^2 \right]^{1/2}}{\delta\omega_x + \delta\omega_y - V_x - V_y}$$



EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (11/35)

- **Due to its infinite tails, the Lorentzian frequency distribution underestimate 2 important points**
  - **The 1st is the effect of the real betatron frequency shift, as already discussed in the uncoupled case**
  - **As a 2nd point, which is in fact closely related to the 1st, it will be found that too strong coupling can be detrimental and may shift the coherent frequency outside the spectrum and thus again prevent Landau damping. To study these two effects, consider elliptical spectra**

$$\left\{ \omega_c - \omega_{x0} - 2U_x - j\left[\sqrt{\Delta\omega_x^2 - \left(\omega_c - \omega_{x0}\right)^2} - 2V_x\right]\right\} \times$$

$$\left\{ \omega_c - \omega_{y0} - l\Omega_0 - 2U_y - j\left[\sqrt{\Delta\omega_y^2 - \left(\omega_c - \omega_{y0} - l\Omega_0\right)^2} - 2V_y\right]\right\} = \frac{\left|\underline{\hat{K}}_0\left(l\right)\right|^2 R^4\,\Omega_0^4}{\omega_{x0}\,\omega_{y0}}$$

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (12/35)

- **Consider 1st the case with no horizontal frequency spread and no vertical wake field**

$$\left[\omega_c - \omega_{x0} - U_x + j V_x\right] \times$$
$$\left[\omega_c - \omega_{y0} - l\,\Omega_0 - j\sqrt{\Delta\omega_y^2 - \left(\omega_c - \omega_{y0} - l\,\Omega_0\right)^2}\,\right] = \frac{\left|\underline{\hat{K}}_0(l)\right|^2 R^4\,\Omega_0^4}{2\,\omega_{x0}\,\omega_{y0}}$$

Let's define $$\kappa = \frac{\left|\underline{\hat{K}}_0(l)\right|^2 R^4\,\Omega_0^4}{\Delta\omega_y^2\,\omega_{x0}\,\omega_{y0}}$$

For $\kappa = 1$

$$\omega_c = \omega_{x0} + U_x - j V_x - \frac{\Delta\omega_y^2}{4}\,\frac{\left(\omega_{y0} + l\,\Omega_0 - \omega_{x0} - U_x - j V_x\right)}{\left(\omega_{y0} + l\,\Omega_0 - \omega_{x0} - U_x\right)^2 + V_x^2}$$

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (13/35)

=> Stability criterion $\Delta\omega_y \geq 2\sqrt{\Omega_0^2 \left(Q_h - Q_v - l \right)^2 + V_x^2}$

For $\kappa \neq 1$

$$\left[\omega_c - \omega_{x0} - U_x + j V_x \right] \times$$
$$\left[\omega_c + \frac{U_x + \omega_{x0} - \kappa \left(\omega_{y0} + l \Omega_0 \right)}{\kappa - 1} - j \frac{V_x}{\kappa - 1} \right] = \frac{\left| \hat{\underline{K}}_0 \left(l \right) \right|^2 R^4 \, \Omega_0^4}{4 \, \omega_{x0} \, \omega_{y0}} \left(\frac{\kappa}{\kappa - 1} \right)$$

The necessary condition for stability ($-V_x + \frac{V_x}{\kappa - 1} \geq 0$ **) leads to**

$$1 \leq \kappa \leq 2$$

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (14/35)

The imaginary parts of the 2 coherent oscillation frequencies are given by

$$\mathrm{Im}\left(\omega_{c1}\right) = -V_x + \frac{V_x\,\kappa}{2\left(\kappa-1\right)} C\left(a',\delta'\right)$$

$$\mathrm{Im}\left(\omega_{c2}\right) = \frac{V_x}{\kappa-1} - \frac{V_x\,\kappa}{2\left(\kappa-1\right)} C\left(a',\delta'\right)$$

Same sharing function as before (for Lorentzian)

$$a' = \frac{\Delta\omega_y}{2V_x}\sqrt{\kappa-1}$$

$$\delta' = \frac{\Omega_0\left|Q_h - Q_v - l\right|}{V_x}\left(\frac{\kappa-1}{\kappa}\right)$$

=> Stability criterion

$$\boxed{\left|Q_h - Q_v - l\right| \le \frac{1}{\Omega_0}\left(\frac{\Delta\omega_y^2}{4}\kappa^2 - \Delta\omega_y^2\,\kappa + \Delta\omega_y^2 - V_x^2 + \frac{4V_x^2}{\kappa} - \frac{4V_x^2}{\kappa^2}\right)^{1/2}}$$

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (15/35)

- If $\Delta\omega_y < V_x$, then it is impossible to stabilise the beam by coupled Landau damping: there is not enough Landau damping which can be transferred to the unstable plan
- The minimum frequency spread that can stabilise the beam is

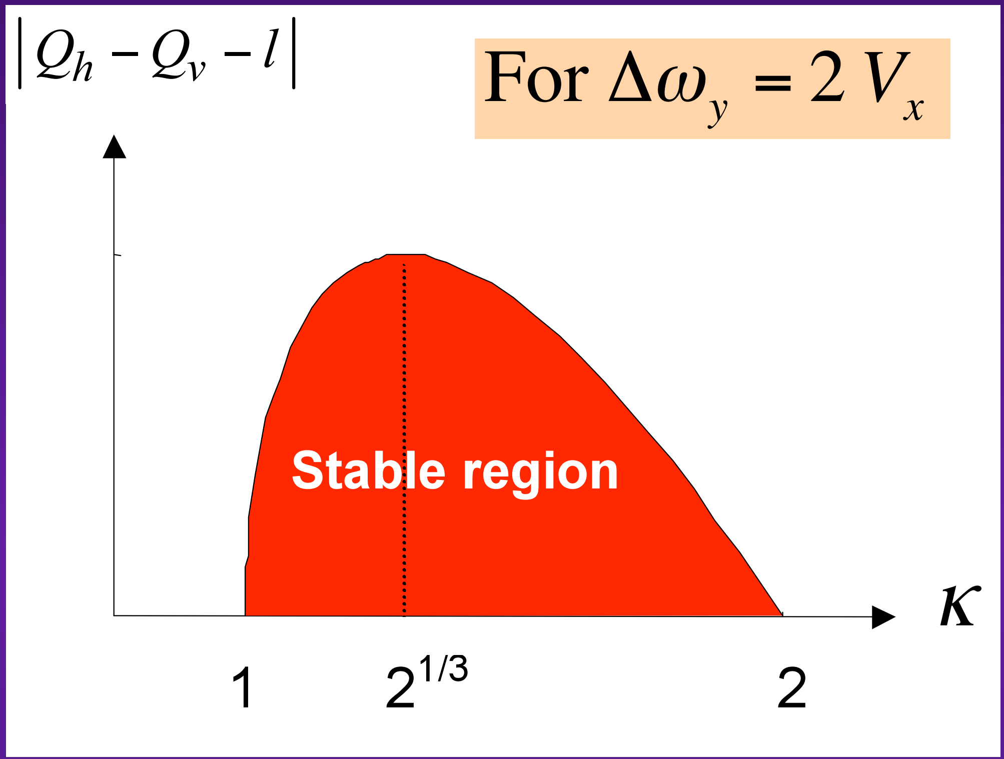
$$\Delta\omega_y = V_x$$

In this case, there is only one condition for stability which is

$$Q_h - Q_v - l = 0 \qquad \kappa = 2$$

- If $\Delta\omega_y > V_x$, we can plot the curve describing the stability boundary, which takes different forms according to the value of $\Delta\omega_y$

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (16/35)



The "optimum" coupling κ leads to the maximum of tune split tolerable for stability. It is obtained for $\kappa = 2^{1/3}$ and the corresponding maximum tune split is

$$\left|Q_h - Q_v - l\right|_{\max} = \left(\sqrt{3}\,V_x / \Omega_0\right)\sqrt{1 + 2^{2/3}\left(1 - 2^{2/3}\right)}$$

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (17/35)

- **Consider now (as a 2nd case) the case with no horizontal frequency spread only**

$$\left[\omega_c - \omega_{x0} - U_x + j\,V_x\right] \times$$

$$\left\{\omega_c - \omega_{y0} - l\,\Omega_0 - 2U_y - j\left[\sqrt{\Delta\omega_y^2 - \left(\omega_c - \omega_{y0} - l\,\Omega_0\right)^2} - 2V_y\right]\right\} = \frac{\left|\hat{\underline{K}}_0(l)\right|^2 R^4\,\Omega_0^4}{2\,\omega_{x0}\,\omega_{y0}}$$

Putting ω_c real and equating the real and imaginary parts separately, yields the conditions at the stability limit

$$\left(\omega_c - \omega_{x0} - U_x\right)\left(\omega_c - \omega_{y0} - l\,\Omega_0 - 2U_y\right)$$

$$-V_x\left[2V_y - \sqrt{\Delta\omega_y^2 - \left(\omega_c - \omega_{y0} - l\,\Omega_0\right)^2}\right] = \frac{\left|\hat{\underline{K}}_0(l)\right|^2 R^4\,\Omega_0^4}{2\,\omega_{x0}\,\omega_{y0}} \qquad \text{(i)}$$

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (18/35)

$$V_x\left(\omega_c-\omega_{y0}-l\Omega_0-2U_y\right)+\left[2V_y-\sqrt{\Delta\omega_y^2-\left(\omega_c-\omega_{y0}-l\Omega_0\right)^2}\,\right]\left(\omega_c-\omega_{x0}-U_x\right)=0 \qquad \text{(ii)}$$

Let's consider the case where

$$\omega_{x0}+U_x=\omega_{y0}+l\Omega_0+2U_y \qquad \text{(iii)}$$

Then (ii) is verified if

$$\omega_c=\omega_{x0}+U_x \qquad \text{(iv)}$$

or

$$\sqrt{\Delta\omega_y^2-\left(\omega_c-\omega_{y0}-l\Omega_0\right)^2}=V_x+2V_y \qquad \text{(v)}$$

If (iv) is fulfilled then (i) becomes

$$\frac{\left|\hat{K}_0(l)\right|^2 R^4\,\Omega_0^4}{2\,\omega_{x0}\,\omega_{y0}}=V_x\left[\sqrt{\Delta\omega_y^2-\left(\omega_c-\omega_{y0}-l\Omega_0\right)^2}-2V_y\right] \qquad \text{(vi)}$$

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (19/35)

If (v) is verified then

$$\left(\omega_c - \omega_{y0} - l\,\Omega_0\right)^2 = \Delta\omega_y^2 - \left(V_x + 2V_y\right)^2 \quad \text{(vii)}$$

Besides, (i) yields

$$\left(\omega_c - \omega_{x0} - U_x\right)^2 = \frac{\left|\hat{\underline{K}}_0(l)\right|^2 R^4\,\Omega_0^4}{2\,\omega_{x0}\,\omega_{y0}} - V_x^2 \quad \text{(viii)}$$

Using (iii), (vii) and (viii), one obtains (ix)

$$\omega_c = \omega_{y0} + l\,\Omega_0 + U_y + \frac{1}{4U_y}\left(\Delta\omega_y^2 - \left(V_x + 2V_y\right)^2 - \frac{\left|\hat{\underline{K}}_0(l)\right|^2 R^4\,\Omega_0^4}{2\,\omega_{x0}\,\omega_{y0}} + V_x^2\right)$$

Equating $\left(\omega_c - \omega_{y0} - l\,\Omega_0\right)^2$ in (ix) and (vii) gives a second order equation in $\Delta\omega_y^2 - \left(V_x + 2V_y\right)^2$. This can be solved in $\Delta\omega_y^2$

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (20/35)

$$\Delta\omega_y^2 = 4\,U_y^2 + 4\,V_y^2 + 4\,V_x\,V_y + \frac{\left|\underline{\hat{K}}_0(l)\right|^2 R^4\,\Omega_0^4}{2\,\omega_{x0}\,\omega_{y0}} \pm 4\,U_y\left(\frac{\left|\underline{\hat{K}}_0(l)\right|^2 R^4\,\Omega_0^4}{2\,\omega_{x0}\,\omega_{y0}} - V_x^2\right)^{1/2}$$

Using (vi), the following equation is obtained

$$\left(\Delta\omega_y^2 - 4\,U_y^2\right)^2 - 2V_x\left(\Delta\omega_y^2 - 4\,U_y^2\right)^{3/2} + \left[V_x^2 - 4\,V_y\left(V_x + 2V_y\right)\right]\left(\Delta\omega_y^2 - 4\,U_y^2\right)$$
$$+4\,V_x\left[V_y\left(V_x + 2V_y\right) - 4\,U_y^2\right]\left(\Delta\omega_y^2 - 4\,U_y^2\right)^{1/2} + 4\,V_y^2\left(V_x + 2V_y\right)^2 + 16\,U_y^2\,V_x\left(V_x + 2V_y\right) = 0$$

One has to solve this equation to find the two-dimensional criterion for $\Delta\omega_y$. Let's consider the practical case where $|U| >> V$, then the stability criterion reduces approximately to

$$\Delta\omega_y \geq \sqrt{4\,U_y^2 + (16\,V_x\,U_y^2)^{2/3}}$$

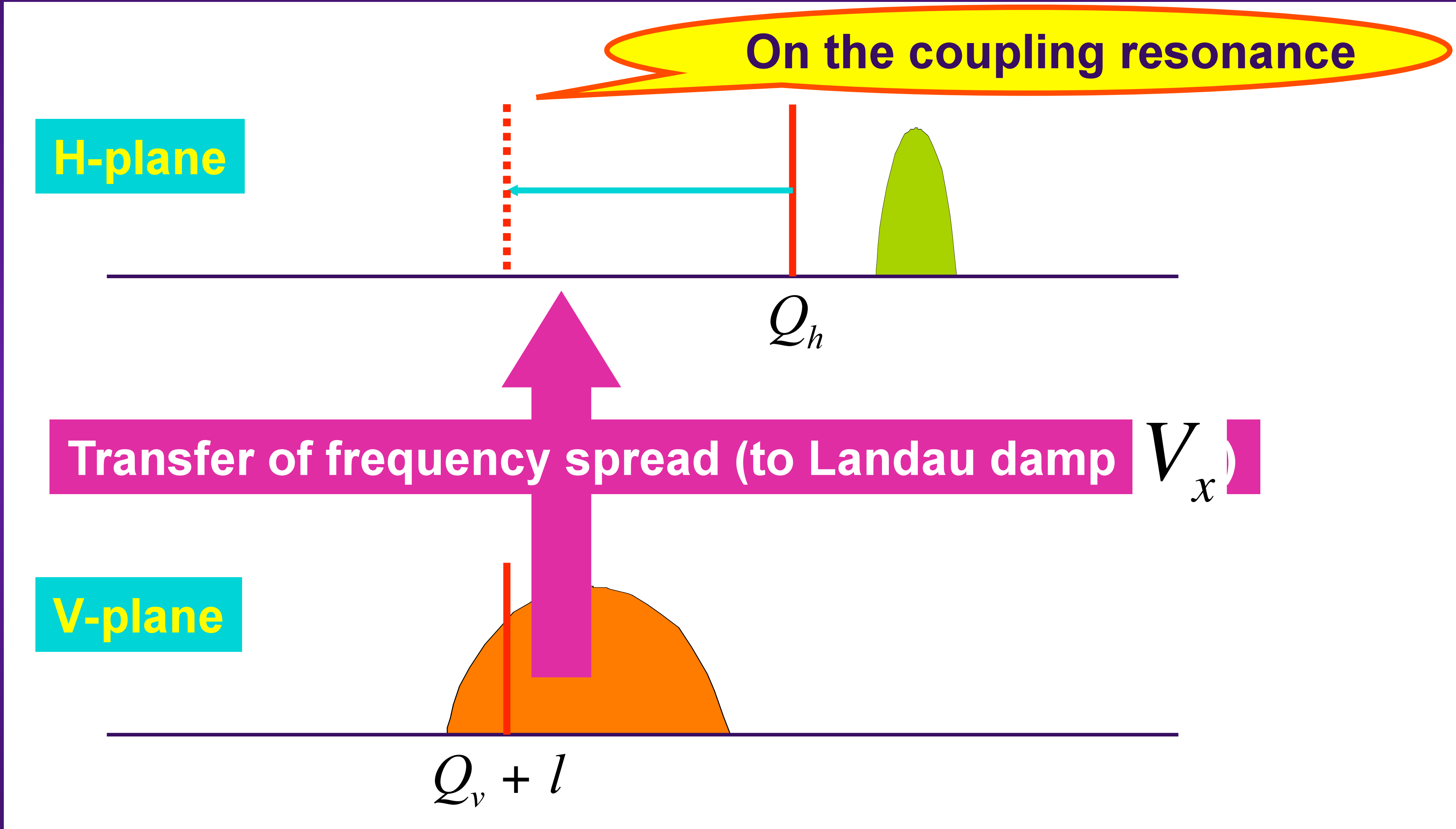
EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (21/35)

Since $\left| U \right| >> V$, a rough criterion is therefore just

$$\boxed{\Delta\omega_y \geq 2 \left| U_y \right|}$$

This is a very simple and interesting result which shows all the effectiveness of coupling in machines where $\left| U \right| >> V$. At one dimension, one has to compensate U and V , for both planes separately. If one uses coupled Landau damping, the main part of the job, the cancellation of the effect of U_x , is done by coupling. Then it roughly remains to Landau damp V_x. Loosely speaking, one plane is thus stabilised by Landau damping and the other one is stabilised by coupling

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (22/35)



EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (23/35)

- **Consider now (as a 3rd and last case) the case with the same distribution in both planes**

$$\Delta\omega_x = \Delta\omega_y = \Delta\omega \qquad \omega_{x0} = \omega_{y0} + l\,\Omega_0 = \omega_0$$

$$\left[\omega_c - \omega_0 - j\sqrt{\Delta\omega^2 - \left(\omega_c - \omega_0\right)^2} + 2\left(-U_x + j\,V_x\right)\right] \times$$

$$\left[\omega_c - \omega_0 - j\sqrt{\Delta\omega^2 - \left(\omega_c - \omega_0\right)^2} + 2\left(-U_y + j\,V_y\right)\right] = \frac{\left|\underline{\hat{K}}_0\left(l\right)\right|^2 R^4\,\Omega_0^4}{\omega_{x0}\,\omega_{y0}}$$

This is a 2nd order equation in whose solutions are

$$X = \omega_c - \omega_0 - j\sqrt{\Delta\omega^2 - \left(\omega_c - \omega_0\right)^2}$$

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (24/35)

$$X_{1,2} = \left(U_x + U_y \pm \frac{1}{2} \frac{B}{|B|} \sqrt{\frac{A + \sqrt{A^2 + B^2}}{2}} \right) + j \left(-V_x - V_y \pm \frac{1}{2} \sqrt{\frac{-A + \sqrt{A^2 + B^2}}{2}} \right)$$

$$A = \frac{4 \left| \underline{\hat{K}}_0(l) \right|^2 R^4 \, \Omega_0^4}{\omega_{x0} \, \omega_{y0}} + 4 \left(U_x - U_y \right)^2 - 4 \left(V_x - V_y \right)^2$$

$$B = -8 \left(U_x - U_y \right) \left(V_x - V_y \right)$$

=>

$$\omega_{c1,2} = \omega_0 + X_{\mathrm{R}1,2} \frac{\Delta\omega^2 + X_{\mathrm{R}1,2}^2 + X_{\mathrm{I}1,2}^2}{2\left[X_{\mathrm{R}1,2}^2 + X_{\mathrm{I}1,2}^2 \right]} - j \, X_{\mathrm{I}1,2} \frac{\Delta\omega^2 - X_{\mathrm{R}1,2}^2 - X_{\mathrm{I}1,2}^2}{2\left[X_{\mathrm{R}1,2}^2 + X_{\mathrm{I}1,2}^2 \right]}$$

$$X_{\mathrm{R}1,2} = \mathrm{Re}\left(X_{1,2} \right) \qquad X_{\mathrm{I}1,2} = \mathrm{Im}\left(X_{1,2} \right)$$

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (25/35)

If $U_x = U_y = U$ **and** $V_x = V_y = V$

$$\Delta\omega \geq 2\sqrt{\left(\left|U\right| + \frac{\left|\hat{\underline{K}}_0\left(l\right)\right| R^2\,\Omega_0^2}{2\sqrt{\omega_{x0}\,\omega_{y0}}}\right)^2 + V^2}$$

This is a very simple and interesting result which shows that if the coupling is too strong the coherent frequency is shifted outside the incoherent frequency spread and Landau damping is lost

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (26/35)

- Therefore, linear coupling can have beneficial or detrimental effects

- In addition, particular care should be also paid to the transverse emittances as in the case of non round beams a sharing and/or transfer of emittances can take place

- To see this, let's consider the case near the linear coupling resonance $\Delta = Q_y - Q_x = 0$. In this case the following equations are obtained (see previous slides, but now without taking into account the wake fields)

$$\frac{d^2\eta}{d\phi^2} + Q_x^2\,\eta = R^2\,\underline{K}_0\,\zeta \qquad \frac{d^2\zeta}{d\phi^2} + Q_y^2\,\zeta = R^2\,\underline{K}_0\,\eta$$

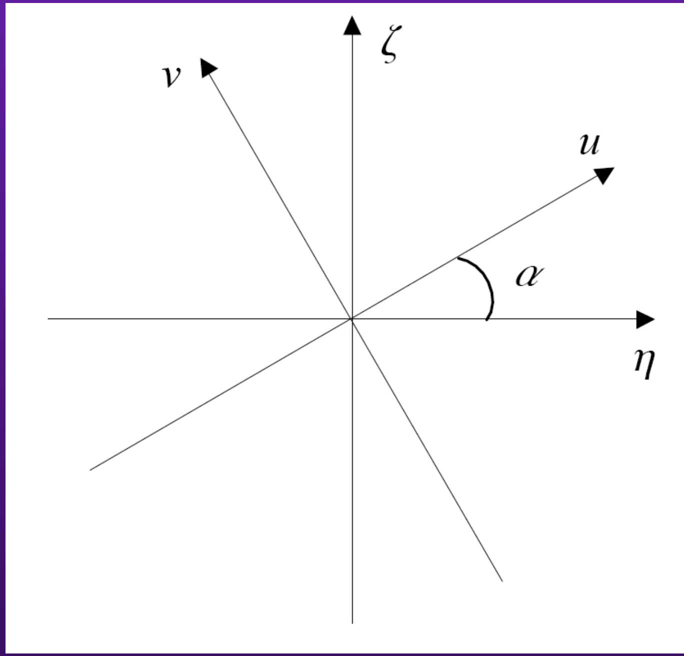
EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (27/35)

- **In the absence of coupling, the solutions of the homogeneous equations are given by**

$$\eta = \eta_0 \, e^{j Q_x \phi} \qquad \zeta = \zeta_0 \, e^{j Q_y \phi}$$

- **In the presence of coupling, the coupled equations can be solved by searching the normal (i.e. decoupled) modes (*u*, *v*), which are linked to (η, ζ) by a simple rotation**

$$\begin{pmatrix} \eta \\ \zeta \end{pmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$



EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (28/35)

- **The equations of the 2 normal modes are**

$$\frac{d^2u}{d\phi^2} + Q_u^2\, u = 0 \qquad \frac{d^2v}{d\phi^2} + Q_v^2\, v = 0$$

with (assuming small tune shifts)

$$Q_u = Q_x - \frac{|C|}{2}\tan\alpha \qquad Q_v = Q_y + \frac{|C|}{2}\tan\alpha$$

$$\tan(2\alpha) = \frac{|C|}{\Delta}$$

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (29/35)

- **The solutions are given by**

$$u = \eta_0 \, e^{\,j Q_u \phi} \qquad \qquad v = \zeta_0 \, e^{\,j Q_v \phi}$$

- **Initially, the 2 planes are considered decoupled (there is a constant coupling $|C|$ in time, but $\Delta >> |C|$)**

$$\eta = \eta_0 \, e^{\,j Q_u \phi} \cos\alpha - \zeta_0 \, e^{\,j Q_v \phi} \sin\alpha$$

$$\zeta = \eta_0 \, e^{\,j Q_u \phi} \sin\alpha + \zeta_0 \, e^{\,j Q_v \phi} \cos\alpha$$

- **By definition, the horizontal and vertical "single-particle" emittances are given by**

$$\varepsilon_x^{sp} = |\eta|^2 \qquad \qquad \varepsilon_y^{sp} = |\zeta|^2$$

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (30/35)

which yields

$$\varepsilon_x^{sp} = \varepsilon_{x0}^{sp} \cos^2\alpha + \varepsilon_{y0}^{sp} \sin^2\alpha - \sqrt{\varepsilon_{x0}^{sp}\, \varepsilon_{y0}^{sp}} \sin(2\alpha) \cos\left[\left(Q_v - Q_u \right) \phi \right]$$

$$\varepsilon_y^{sp} = \varepsilon_{x0}^{sp} \sin^2\alpha + \varepsilon_{y0}^{sp} \cos^2\alpha + \sqrt{\varepsilon_{x0}^{sp}\, \varepsilon_{y0}^{sp}} \sin(2\alpha) \cos\left[\left(Q_v - Q_u \right) \phi \right]$$

where $\varepsilon_{x0}^{sp} = \eta_0^2$ **and** $\varepsilon_{y0}^{sp} = \zeta_0^2$ **are the initial uncoupled single-particle transverse emittances**

- **It can be seen that in the presence of linear coupling the sum of the single-particle emittances is always conserved**

$$\varepsilon_x^{sp} + \varepsilon_y^{sp} = \varepsilon_{x0}^{sp} + \varepsilon_{y0}^{sp}$$

- **If one now wants to look at the rms emittances (i.e. of the beam), one has to average over time (which is equivalent to an average over *Φ*), and over the particles in the beam**

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (31/35)

- **The 1st gives**

$$\overline{\varepsilon_x^{sp}} = \varepsilon_{x0}^{sp} \cos^2 \alpha + \varepsilon_{y0}^{sp} \sin^2 \alpha \qquad \overline{\varepsilon_y^{sp}} = \varepsilon_{x0}^{sp} \sin^2 \alpha + \varepsilon_{y0}^{sp} \cos^2 \alpha$$

Note that on the coupling resonance, $Q_v - Q_u = |C|$. The oscillation period of the cosine terms of the previous page is thus $T_\phi = 2\pi / |C|$

If $|C|$ is infinitely small, then an infinitely long time is needed to cross the resonance to average this term to 0. The

- **The 2nd gives**

$$\varepsilon_x = \varepsilon_{x0} \cos^2 \alpha + \varepsilon_{y0} \sin^2 \alpha \qquad \varepsilon_y = \varepsilon_{x0} \sin^2 \alpha + \varepsilon_{y0} \cos^2 \alpha$$

with $\varepsilon_x = \left\langle \overline{\varepsilon_x^{sp}} \right\rangle$ $\quad \varepsilon_y = \left\langle \overline{\varepsilon_y^{sp}} \right\rangle$ $\quad \varepsilon_{x0} = \left\langle \varepsilon_{x0}^{sp} \right\rangle$ $\quad \varepsilon_{y0} = \left\langle \varepsilon_{y0}^{sp} \right\rangle$

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (32/35)

- **Using the fact that**

$$\cos\left(2\alpha\right) = \cos\left[\arctan\left(\frac{|C|}{\Delta}\right)\right] = \left(1 + \frac{|C|^2}{\Delta^2}\right)^{-1/2}$$

one can show that

$$\sin^2\alpha = \frac{|C|^2/2}{\Delta^2 + |C|^2 + \Delta\sqrt{\Delta^2 + |C|^2}}$$

and therefore

$$\boxed{\varepsilon_x = \varepsilon_{x0} - \left(\varepsilon_{x0} - \varepsilon_{y0}\right)\frac{|C|^2/2}{\Delta^2 + |C|^2 + \Delta\sqrt{\Delta^2 + |C|^2}}}$$

$$f\left(|C|, \Delta\right) = \frac{|C|^2/2}{\Delta^2 + |C|^2 + \Delta\sqrt{\Delta^2 + |C|^2}}$$

$$\boxed{\varepsilon_y = \varepsilon_{y0} + \left(\varepsilon_{x0} - \varepsilon_{y0}\right)\frac{|C|^2/2}{\Delta^2 + |C|^2 + \Delta\sqrt{\Delta^2 + |C|^2}}}$$

$$\frac{\partial f}{\partial \Delta}\left(|C|, \Delta = 0\right) = -\frac{1}{2|C|}$$

EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (33/35)

- One sees that in the presence of very small coupling, i.e. $\Delta >> |C|$, the transverse emittances are given by

$$\varepsilon_x = \varepsilon_{x0} \qquad \varepsilon_y = \varepsilon_{y0}$$

- As coupling increases, the sharing of the emittances increases and reaches its maximum value for full coupling, where the emittances are given by

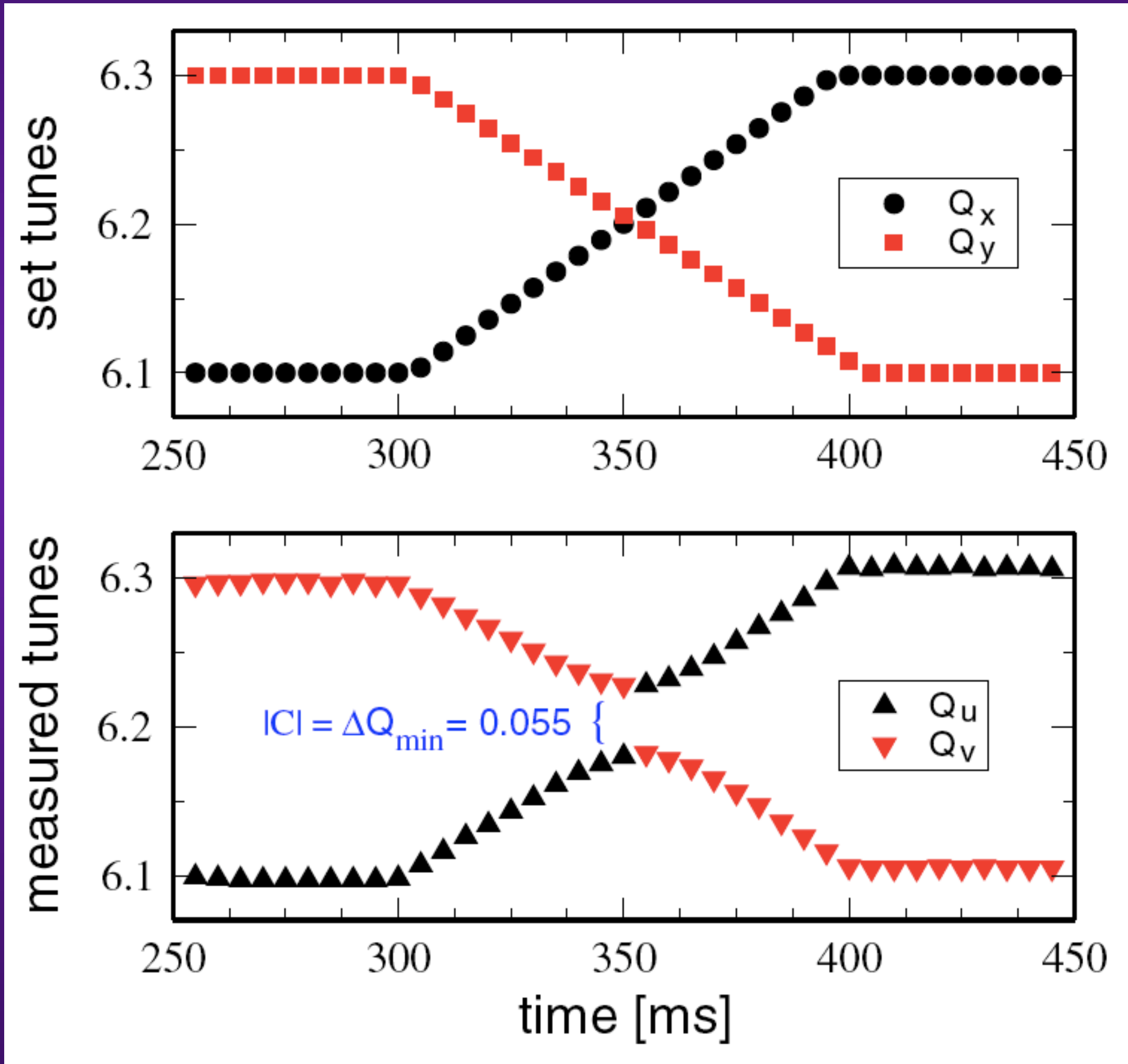
$$\varepsilon_x = \varepsilon_y = \frac{\varepsilon_{x0} + \varepsilon_{y0}}{2}$$

- In the presence of very small coupling again, after the resonance crossing, i.e. $-\Delta >> |C|$, one has

$$\varepsilon_x = \varepsilon_{y0} \qquad \varepsilon_y = \varepsilon_{x0}$$

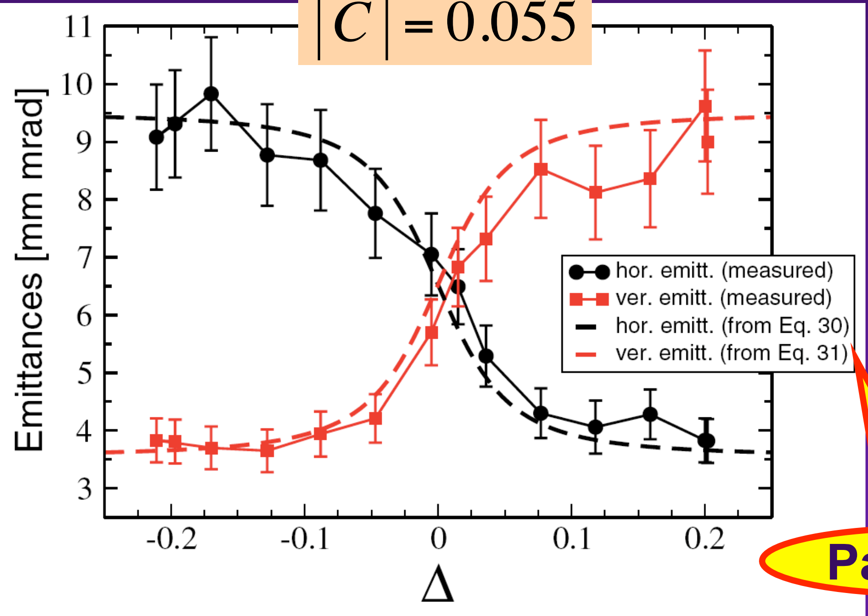
EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (34/35)

- **Measurements of emittance exchange performed in the CERN PS near** $Q_x = Q_y$

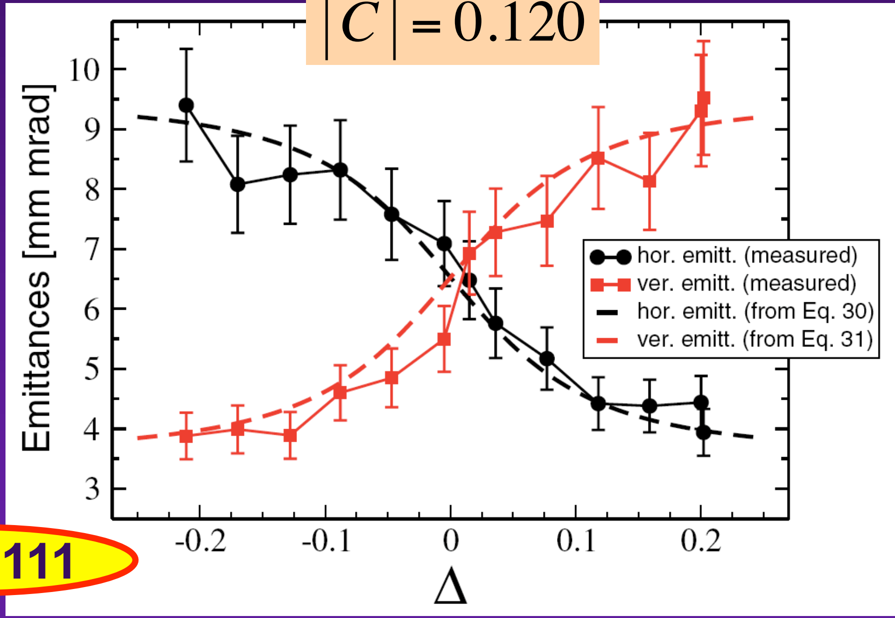


EFFECT OF LINEAR COUPLING BETWEEN THE TRANSVERSE PLANES (35/35)

$\left| C \right| = 0.055$



$\left| C \right| = 0.120$



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$\left| C \right| = 0.016$