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Negative Mass Effect in Nonlinear Oscillatory System and its Influence on Stability of Coherent Betatron Oscillations*

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Abstract

The paper deals with stability of coherent betatron oscillations of the beam in a storage ring with the account of the guide field nonlinearity and coherent and/or incoherent tuneshift of the betatron oscillations.

The stability of coherent betatron oscillations is determined by an essentially nonlinear effect, an analogue of the negative mass effect in longitudinal motion. The stability conditions are found, a) determined by relation of the *sign* of the cubic nonlinearity of the guide field and the sign of the collective force; b) dependent on the collective betatron tuneshift and the betatron tune spread in the beam.

1 Introduction

In studies of stability of coherent beam motion in accelerators and storage rings, influence of the cubic nonlinearity of the guide field is conventionally accounted in the tune spread which determines the threshold current for the instability onset [1,2]. In this approach only the *absolute value* of the cubic nonlinearity is important.

In the present paper we show that an essentially nonlinear effect, similar to the negative mass effect in longitudinal motion, plays a decisive role in analysis of the coherent betatron motion, fast damping, etc. With the account of this effect, the *sign* of the cubic nonlinearity is crucial.

Experimental results from the VEPP-3 storage ring are well explained with this "negative mass effect in the betatron phase space". Details in observed decoherence signals on VEPP-1 [3] and fast damping signals on SPEAR [4] also find an explanation in the framework of this model.

2 Hamiltonian of Coherent Betatron Oscillations

Consider one-dimensional (e.g., vertical) coherent betatron oscillations of the beam in a storage ring, and take into account a weak nonlinearity of the guide magnetic field, $f(z,\theta)$, and a weak transverse collective force, $\Phi(\theta)(z-y)$, directed to (from) the beam centroid, where y is the position of the beam charge centroid, z is the coordinate of an individual particle and θ is the machine azimuth. The equations of motion,

$$\begin{aligned} z'' + g(\theta)\, z &= f(z,\theta) + \Phi(\theta)\,(z-y)\,, \\ y'' + g(\theta)\, y &= f(y,\theta)\,, \end{aligned} \tag{1}$$

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after the standard ansatz for the dynamical variables,

$$\begin{aligned} y &= b\,w(\theta)\,\cos\left(\nu\,\theta + \chi(\theta) + \varphi\right)\,, \\ z &= a\,w(\theta)\,\cos\left(\nu\,\theta + \chi(\theta) + \psi\right)\,, \end{aligned}$$

yield the set of equations

$$\begin{aligned} b' &= -f(y,\theta)\,w(\theta)\sin\left(\nu\,\theta + \chi(\theta) + \varphi\right)\,, \\ \varphi' &= -\frac{1}{b}f(y,\theta)\,w(\theta)\cos\left(\nu\,\theta + \chi(\theta) + \varphi\right)\,, \\ a' &= -\left[f(z,\theta) + \Phi(\theta)\,(z-y)\right]w(\theta)\sin\left(\nu\,\theta + \chi(\theta) + \psi\right)\,, \\ \psi' &= -\frac{1}{a}\left[f(z,\theta) + \Phi(\theta)\,(z-y)\right]w(\theta)\cos\left(\nu\,\theta + \chi(\theta) + \psi\right)\,. \end{aligned} \tag{2}$$

We defined the focusing function of the lattice $g(\theta)$, the nominal betatron tune ν, whilst $w(\theta)e^{i\chi(\theta)}$ is the complex Floquet function.

For small f and Φ, the amplitudes and phases (b,φ), (a,ψ) in Eqs. (2) can be regarded as slow variables. Apart from a resonance, $\nu \neq p/q$, where p, q are integers, we can average Eqs. (2) over the fast betatron phase and azimuth to obtain shortened equations

$$\begin{aligned} a' &= -\delta\nu\, b\, \sin(\psi - \varphi)\,, \\ \psi' &= \frac{\partial\,\nu}{\partial\,a^2}a^2 + \delta\nu\left[1 - \frac{b}{a}\cos(\psi-\varphi)\right]\,, \\ b' &= 0\,, \\ \varphi' &= \frac{\partial\,\nu}{\partial\,a^2}b^2, \end{aligned} \tag{3}$$

where

$$\delta\nu = -\frac{1}{2}\left\langle w^2(\theta)\Phi(\theta)\right\rangle$$

is the incoherent betatron tuneshift, and

$$\frac{\partial\,\nu}{\partial\,a^2} = \frac{3}{8}\left\langle \mu(\theta)w^4(\theta)\right\rangle$$

is the anharmonicity caused by the cubic nonlinearity in the guide field,

$$f(y,\theta) = -\lambda(\theta)\,z^2 - \mu(\theta)\,z^3 - \varepsilon(\theta)\,z^4 + o(z^5)\,.$$

In the action-angle variables $J \equiv a^2 - b^2$, $\eta \equiv \psi - \varphi$, Eqs. (3) form a canonical set with the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\frac{\partial\,\nu}{\partial\,a^2}J^2 + \delta\nu\,\left(J - 2b\sqrt{b^2+J}\cos\eta\right)\,. \tag{4}$$

The relative phase η is counted from the beam centroid phase, and J is its conjugate momentum. The first term in the Hamiltonian (4) is similar to kinetic energy, with the effective mass $(\partial\nu/\partial a^2)^{-1}$, while the second term is characteristic of a force directed to (from) the beam centroid.

2.1 Stability Condition

For small deviations from the beam centroid, $J \ll b^2$, $\eta \ll 1$, from Eq. (4) follows the linearized Hamiltonian

$$\mathcal{H}_l = \frac{1}{2}\left(\frac{\partial \nu}{\partial a^2} + \frac{\delta\nu}{2b^2}\right) J^2 + \delta\nu\, b^2 \eta^2, \tag{5}$$

which yields linear phase oscillations with the frequency

$$\nu_\phi^2 = \delta\nu \left(\delta\nu + 2b^2 \frac{\partial \nu}{\partial a^2}\right) \tag{6}$$

expressed in units of the revolution frequency ω_0.

Note that the effective mass

$$\mathcal{M} = \left(\frac{\partial \nu}{\partial a^2} + \frac{\delta\nu}{2b^2}\right)^{-1} \tag{7}$$

is modified by the interaction parameter.

Motion around the beam centroid is unstable, if $\nu_\phi^2 < 0$, or, in explicit form:

$$-1 < \frac{\delta\nu}{2b^2 \dfrac{\partial \nu}{\partial a^2}} < 0\,. \tag{8}$$

In other words, the motion is unstable when the sign of the driving force is opposite to the sign of the effective mass $\mathcal{M}$. Converse situation leads to autophasing in the betatron phase space: the driving force will bunch the beam particles around its center; the equilibrium phase will then coincide with the phase of the beam centroid oscillations.

Contrary to the conventional negative mass effect in rotational motion, e.g., in longitudinal motion of particles in a circular accelerator [5], or in motion of satellites around planets [6], here we have the negative mass effect in oscillations.

2.2 Phase Space of the System

Consider the phase space of the system described by Eqs. (3), i.e. the phase-space trajectories $\mathcal{H} = \text{const}$ with the Hamiltonian (5), in the polar coordinates a, η. We distinguish four situations with different location of the fixed points A, B, C of $\mathcal{H}$, depending on the parameter δ defined below.

1) The sign of the effective mass $\mathcal{M}$ is the same as the sign of $\delta\nu$, and opposite to the sign of unperturbed effective mass $\partial\nu/\partial a^2$:

$$\delta \equiv \frac{\delta\nu}{2b^2 \dfrac{\partial \nu}{\partial a^2}} < -1\,.$$

Here the vicinity of the beam centroid, around point B, is stable within the bounds of the *betatron autophasing domain* (BAD) which is drop-shaped and in terms of η it has a width dependent on $\delta\nu$, see Fig. 1d.

2)

$$\delta\nu\mathcal{M} < 0\,, \quad \mathcal{M}\frac{\partial \nu}{\partial a^2} > 0\,, \quad \text{i.e.,} \quad -1 < \delta < 0\,.$$

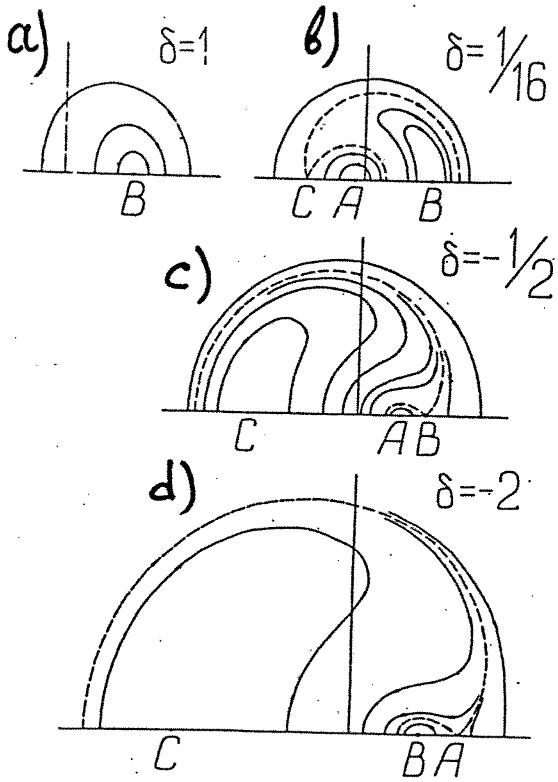

Figure 1: Four different patterns of the negative mass effect in the betatron phase space.

In Fig. 1c we can see the saddle point at B, hence the vicinity of the beam center is unstable. The particles whose initial positions are near point B will loose coherency, the phase η is not bounded.

3)

$$\delta\nu\mathcal{M} > 0\,,\; 0 < \delta < \frac{1}{8}.$$

The beam center B is stable within the crescent-shaped BAD whose width depends on $\delta\nu$, see Fig. 1b.

4) The case $\delta > \frac{1}{8}$ also corresponds to stability of the beam center. However, now all the phase-space trajectories encircle point B which means infinite limits of BAD, see Fig. 1a.

2.3 Beam Size Consideration

As follows from Eq. (8), point B with $J = 0$, $\eta = 0$, can be stable no matter how small $|\delta\nu|$ is. In practice, we should consider the finite beam size σ to decide whether the oscillations of all the particles in the beam can stay coherent with a certain value of $|\delta\nu|$. The necessary condition for the oscillations to stay coherent is that the betatron autophasing domain surrounding B should be wider than the beam size σ. For $0 < \delta \ll 1$, from Eq. (4) we obtain:

$$\sigma \lesssim \left(\delta\nu \Big/ \frac{\partial\,\nu}{\partial\,a^2}\right)^{1/2}. \tag{9}$$

Combining Eqs. (8) and (9), we come to a conclusion: for a beam with transverse size σ oscillations excited by a kick with amplitude b ($b \gg \sigma$) stay coherent (see Figs. 1d or 1b) if

$$\delta\nu < -2\,b^2 \frac{\partial\,\nu}{\partial\,a^2}, \quad \text{or} \quad \delta\nu > \frac{\partial\,\nu}{\partial\,a^2}\sigma^2, \tag{10}$$

the coherent amplitude and the beam size do not vary. In the opposite case, the coherency of the oscillation is lost because of decoherence of oscillations of individual particles of the beam.

In general, the beam shape is not matched with the phase-space ellipse given by Eq. (5), so after the kick excitation the beam size oscillations are possible at the beat frequency $2\nu_\phi$.

Eqs. (6), (8) and (10) relate observable quantities. Provided that the cubic nonlinearity is tunable and the anharmonicity $\partial\nu/\partial a^2$ is measurable, observation of the modulation frequency of the dipole moment of the beam distribution function reveals the collective force directed to (from) the beam center and gives the means to evaluate the collective tuneshift $\delta\nu$. This experimental method yields $\delta\nu$ as a net effect rather than a small correction to the fractional tune $\{\nu\}$ which typically is much greater than $\delta\nu$; this can improve the accuracy of measurements.

3 Extension of the Model for Coherent Tuneshift

Consider now the betatron oscillations of the beam interacting with the machine components, e.g., the vacuum pipe walls, kicker structures, cavities, etc. In the linear approximation, instead of Eq. (1) we write

$$\begin{aligned} z'' + g(\theta)\,z &= f(z,\theta) + A(\theta)\,z + B(\theta)\,y\,, \\ y'' + g(\theta)\,y &= f(y,\theta) + [A(\theta) + B(\theta)]\;y\,, \end{aligned} \tag{11}$$

where A and B describe the collective interaction and are similar to those defined in [7]. Changing the notation,

$$\begin{aligned} g_1 &\equiv g - A - B \\ \Phi &\equiv -B \end{aligned}$$

we reduce the problem to the case considered in Section 1, and apply the same transformations to Eqs. (11). Shortened equations (3), Hamiltonian (4) and conditions (8) and (10) preserve their form with new definitions:

$$\delta\nu = \delta\nu_{\rm ic} - \delta\nu_{\rm c}\,,$$

where

$$\delta\nu_{\rm ic} = -\frac{1}{2}\left\langle w^2(\theta) A(\theta)\right\rangle$$

is the incoherent betatron tuneshift, and

$$\delta\nu_{\rm c} = -\frac{1}{2}\left\langle w^2(\theta)[A(\theta) + B(\theta]\right\rangle$$

is the coherent betatron tuneshift.

For the most cases, in transverse coherent effects increments (or decrements) are much smaller than $\delta\nu$, therefore the amplitude variation is adiabatically slow on the time scale of the considered above effects. That is why conditions (10) are valid to determine existence of dipole modes in most of the coherent effects. In other words, conditions (10) determine the threshold currents for dipole modes in the coherent phenomena.

Experiments at VEPP-3 and SPEAR [4] revealed dependence of the coherent effects on the sign of the cubic nonlinearity, $\partial\nu/\partial a^2$. Conditions (10) provide the explanation of this dependence.

In addition, from Eq. (8) we can conclude that at $\delta\nu \,/ \frac{\partial\nu}{\partial a^2} < 0$ the amplitude growth caused by an instability should cease when the beam centroid amplitudes reach

$$b_{\max} = \left(-\frac{\delta\nu}{2\dfrac{\partial\nu}{\partial a^2}} \right)^{1/2} .$$

This saturation phenomenon, as well as saw-tooth behavior of instability, were observed at many machines.

4 Conclusion

It is worth noting that the above consideration of stability of dipole coherent oscillations with an amplitude much larger than the beam size gives a qualitative approach to the necessary conditions for an instability onset. Assume that the instability is provoked by a coherent seed, i.e., a small fluctuation of the charge density in the beam with non-zero initial dipole moment, and suppose its characteristic size is small as compared with its coherent oscillation amplitude. Then defining the tuneshift due to its dipole moment $\delta\nu_f$ and the total anharmonicity from the guide field and the collective fields $\left(\frac{\partial\nu}{\partial a^2}\right)_\Sigma$, we conclude from Eq. (10) that at $\delta\nu_f \left(\frac{\partial\nu}{\partial a^2}\right)_\Sigma < 0$ the fluctuation is smeared out due to fast decoherence, see Fig. 1c.

Thus, a coherent instability can only develop with one sign of the cubic nonlinearity, while at another sign onset of the instability is prohibited by the negative mass effect in the betatron motion. In particular, this simple rigid beam model provides for visual interpretation of coherent instability conditions, involving the sign of the cubic nonlinearity, as formally obtained in [8].

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