MEASUREMENT OF THE SPACE CHARGE TUNE SPREAD WITH A QUADRUPOLAR PICK-UP: WHERE DOES THE FORMULA COME FROM?

E. Métral

Horizontal space charge tune spread:

$$\Delta Q_{x,\,\mathrm{spread}}^{SC} = -\,\Delta Q_{x,\,\mathrm{linear\ shift}}^{SC} > 0$$

$$\Delta Q_{x,\,\mathrm{spread}}^{SC} = \frac{2\,Q_{x0} - Q_{2x}}{\frac{1}{2}\left(3 - \frac{\sigma_x}{\sigma_x + \sigma_y}\right)}$$

Unperturbed horizontal tune

Horizontal quadrupolar tune

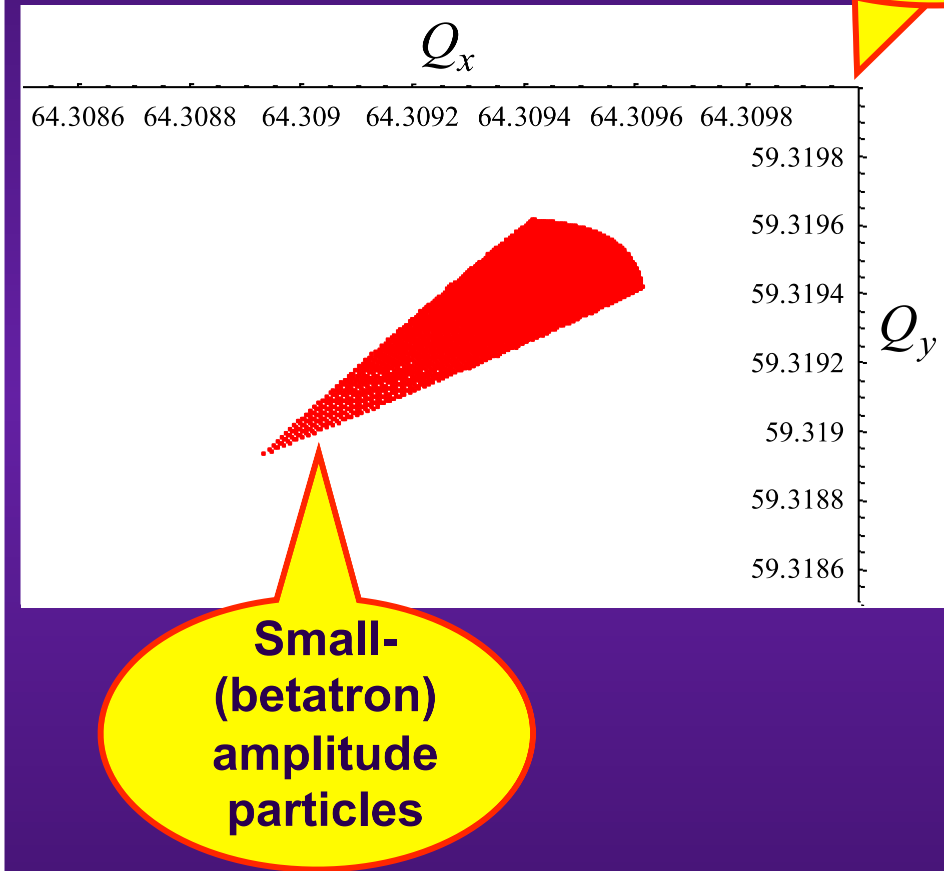
Horizontal rms beam size

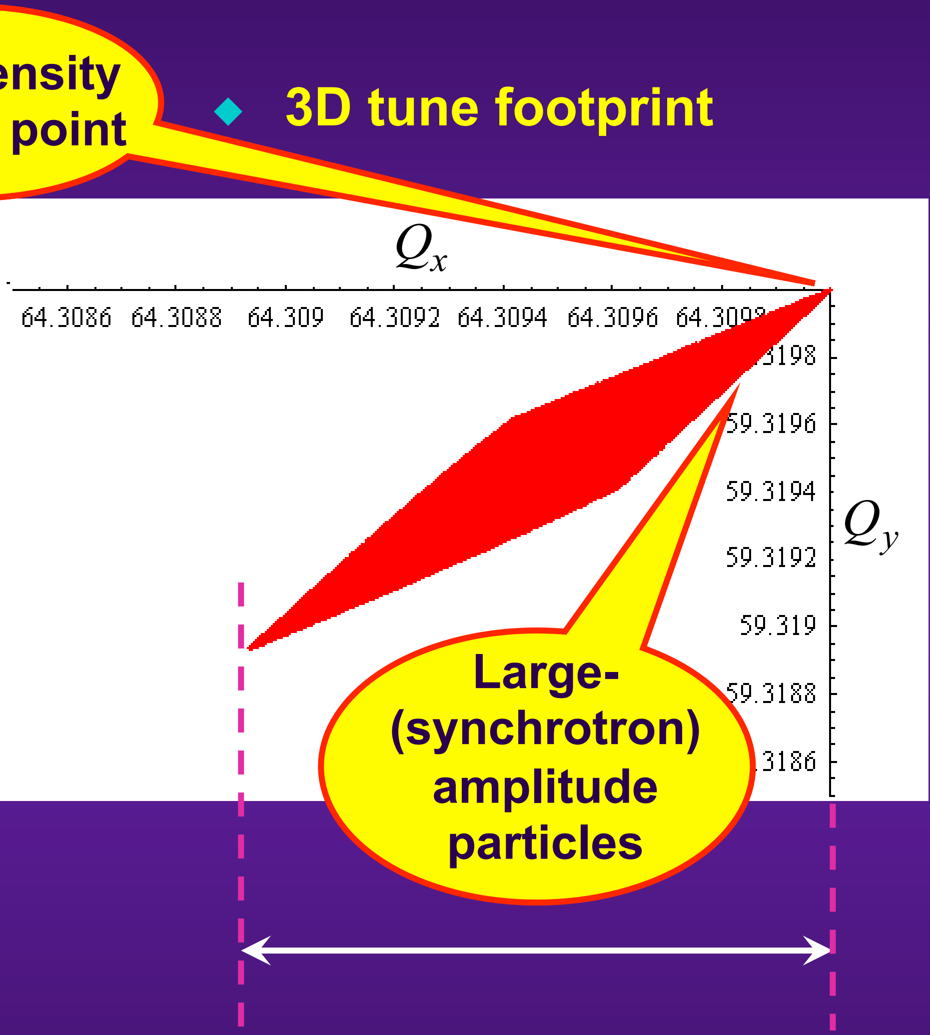
Vertical rms beam size

$$\Delta Q_{y,\,\mathrm{spread}}^{SC} = \frac{2\,Q_{y0} - Q_{2y}}{\frac{1}{2}\left(3 - \frac{\sigma_y}{\sigma_x + \sigma_y}\right)}$$

INTRODUCTION (1/5)

- 2D tune footprint
- 3D tune footprint

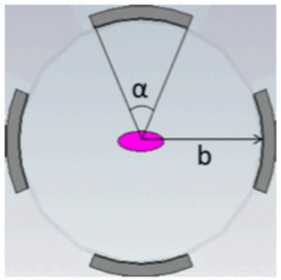


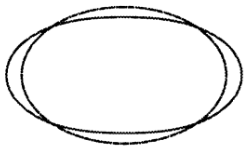


$$\Delta Q^{SC}_{x,\,\text{spread}} = -\,\Delta Q^{SC}_{x,\,\text{linear shift}} > 0$$

Quadrupolar tune spectra

$$P \propto \sigma_x^2 - \sigma_y^2$$





Quadrupolar Pickup (QPU)

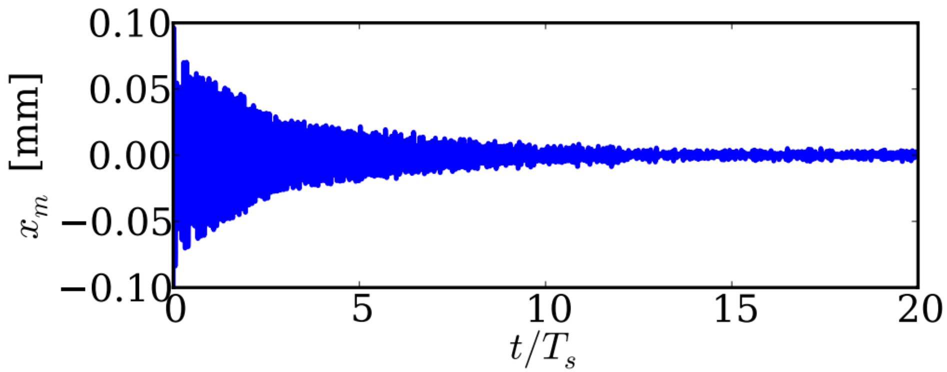
Shift of the quadrupole mode:

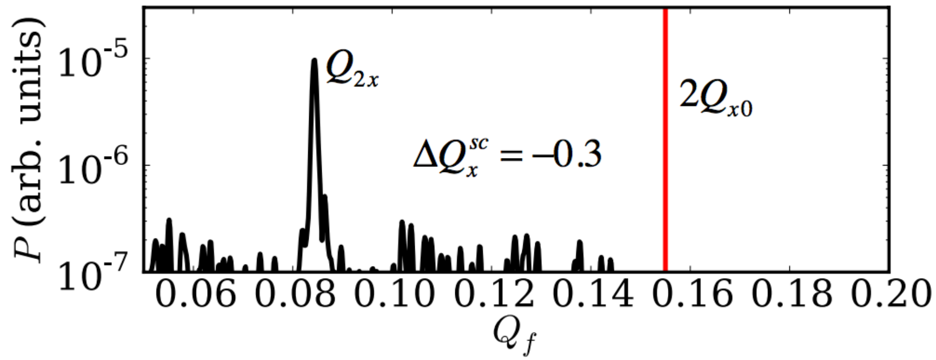
$$Q_{2x} - 2Q_{x0} = -\frac{1}{2}\left(3 - \frac{\sigma_x}{\sigma_x + \sigma_y}\right)\Delta Q_x^{sc}$$

Measurement time: < msec

M. Chanel, Proc. EPAC 1996
R. Bär, I. Hofmann, NIMA 1998

PATRIC simulation: Damping of an initial transverse mismatch oscillation in a bunch with a transverse KV distribution.





Problem: Strong damping of the quadrupolar mode for transverse Gaussian distributions.

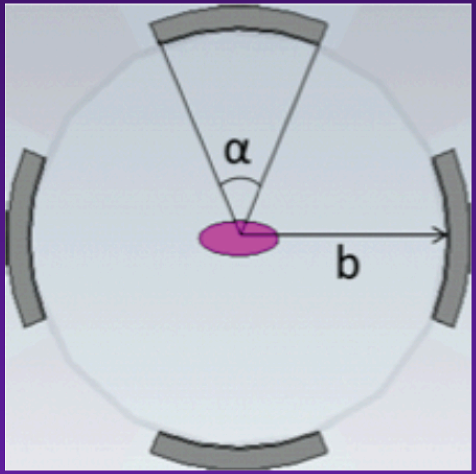
INTRODUCTION (3/5)

- Some past measurements for instance from M. Chanel => Study of beam envelope oscillations by measuring the beam transfer function with quadrupolar pick-up and kicker (https://accelconf.web.cern.ch/accelconf/e96/PAPERS/WEPG/WEP014G.PDF)

- Recent measurements by R. Singh et al. (with Marek Gasior from CERN) => Observations of the quadrupolar oscillations at GSI SIS-18 (http://www.researchgate.net/publication/267195479_OBSERVATIONS_OF_THE_QUADRUPOLAR_OSCILLATIONS_AT_GSI_SIS-18)

=> W. Hardt derived the oscillation frequencies obtained in the presence of space charge forces and gradients errors for elliptical beams (W. Hardt, On the incoherent space charge limit for elliptic beams, CERN/ISR/Int. 300 GS/66.2, 1966)

INTRODUCTION (4/5)



- **Measurement method**
  - The signals of 2 two horizontal electrodes are summed (as well for the vertical electrodes) for the suppression of the transverse dipole modes
  - The two signals obtained are then subtracted to suppress the longitudinal mode (sum signal)
  - The injection mismatch is usually used to excite the beam size oscillations
  - To enhance the signal, an excitation of the beam envelope can also be sent through a quadrupolar kicker (if there is one…)

INTRODUCTION (5/5)

- **Info from Marek Gasior about the QPU**
  - **A frequency domain analysis is made to be able to disentangle between dipolar and quadrupolar frequencies => Only the quadrupolar frequency shift is measured with the QPU**
  - **The transverse beam sizes need to be measured with another equipment**

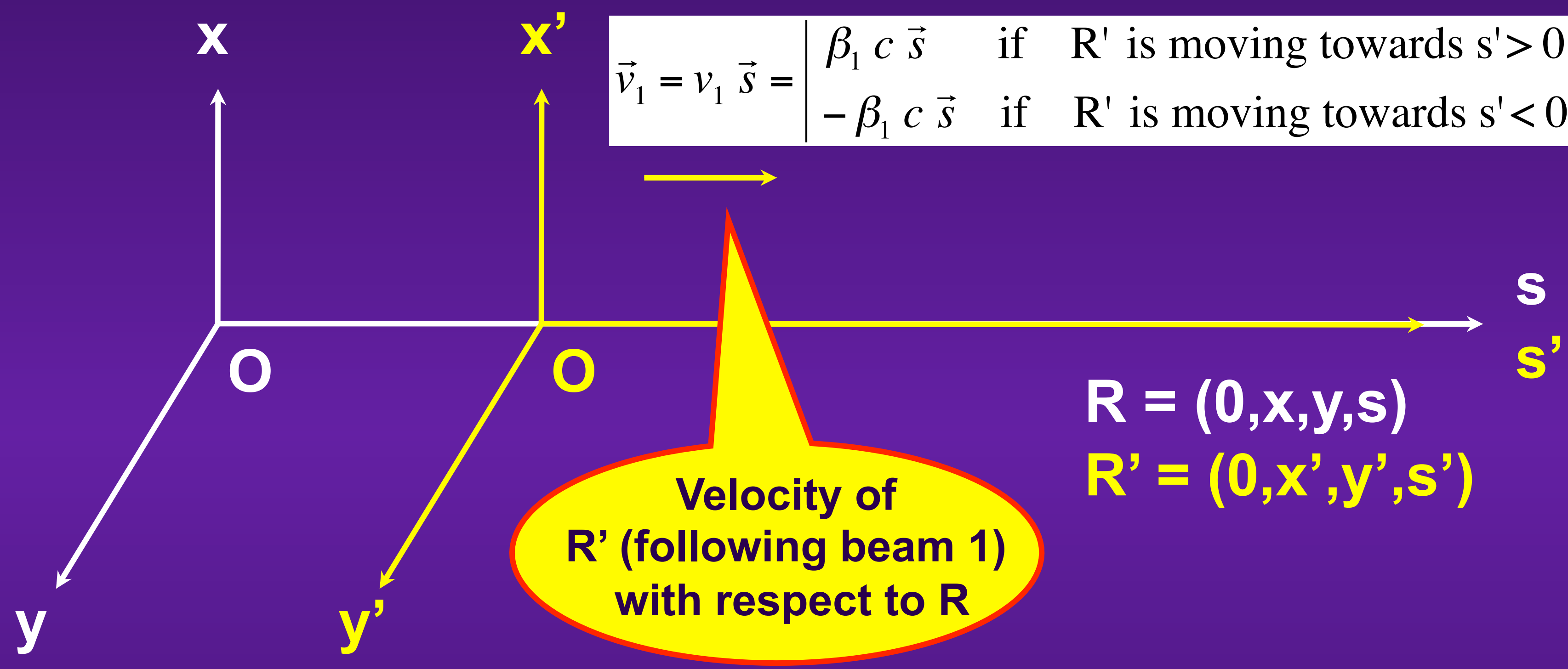
=> Purpose of this talk: re-derive the equation used to deduce the space charge tune spread

See also USPAS2009 course (on space charge and envelope equations) and references therein (http://emetral.web.cern.ch/emetral/)

SPACE CHARGE FORCE (1/5)

- **Relativistic transformation of the ElectroMagnetic (EM) fields**

$$\vec{v}_1 = v_1\,\vec{s} = \begin{vmatrix} \beta_1\, c\, \vec{s} & \text{if} \quad \text{R' is moving towards s'} > 0 \\ -\beta_1\, c\, \vec{s} & \text{if} \quad \text{R' is moving towards s'} < 0 \end{vmatrix}$$



$$E'_x = \gamma_1 \left(E_x - v_1\, B_y \right)$$

$$E'_y = \gamma_1 \left(E_y + v_1\, B_x \right)$$

$$E'_s = E_s \qquad B'_s = B_s$$

$$B'_x = \gamma_1 \left(B_x + \frac{v_1}{c^2}\, E_y \right) \qquad B'_y = \gamma_1 \left(B_y - \frac{v_1}{c^2}\, E_x \right)$$

SPACE CHARGE FORCE (2/5)

- **Lorentz force on the particle 2 moving with velocity** $\vec{v}_2 = v_2 \, \vec{s}$

$$\vec{F} = e \left(\vec{E} + \vec{v}_2 \times \vec{B} \right)$$

- **Beam 1 produces only an electric field in its rest frame R'**

$$B'_x = B'_y = B'_s = 0$$

=> $B_x = -\dfrac{v_1}{c^2} E_y$ $\quad B_y = \dfrac{v_1}{c^2} E_x$ $\quad B_s = 0$

=> $$F_{x,y} = e \, E_{x,y} \left| \begin{array}{ll} \left(1 - \beta_1 \, \beta_2\right) & \text{if 2 moves in same direction as 1} \\ \left(1 + \beta_1 \, \beta_2\right) & \text{if 2 moves in oppo. direction as 1} \end{array} \right.$$

Space charge

Beam beam

SPACE CHARGE FORCE (3/5)

- **Assuming $\beta_1 = \beta_2 = \beta$**

=> $$F_{x,y} = e\, E_{x,y} \left(1 - \beta^2\right) = e\, \frac{E_{x,y}}{\gamma^2}$$

Electric part

Magnetic part

and

$$E'_{x,y} = \frac{E_{x,y}}{\gamma}$$

$$B_x = -\frac{\beta}{c} E_y$$

$$E'_s = E_s$$

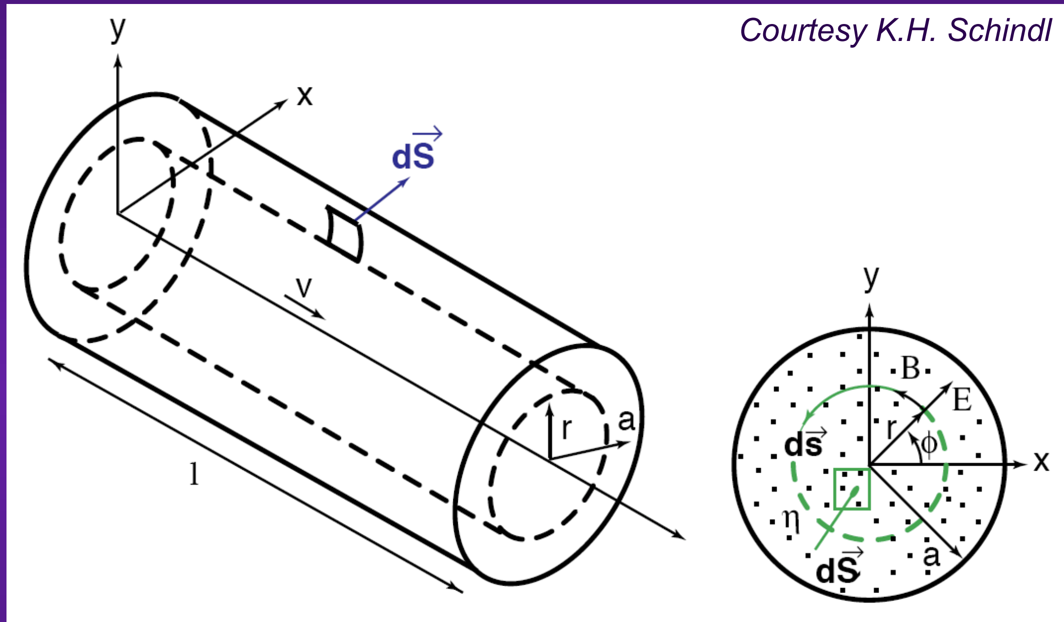
$$B_y = \frac{\beta}{c} E_x$$

$$B'_x = B'_y = B'_s = 0$$

$$B_s = 0$$

SPACE CHARGE FORCE (4/5)

- **EM fields of a cylinder with uniform density (with radius a) inside a beam pipe of radius b**



- **Charge density [C/m³]** $\rho = \dfrac{q}{\pi a^2 l}$

$q = N_b e$

- **Current density [A/m²]** $J = \rho v$

- **Line density [C/m]** $\lambda_0 = \dfrac{q}{l}$

- **Total Current [A]** $I = \lambda_0 v$

SPACE CHARGE FORCE (5/5)

- $$\iiint div\ \vec{E}\ dV = \iint \vec{E}\,.\,d\vec{S} = \frac{1}{\varepsilon} \iiint \rho\ dV$$

=> $$\left| \begin{array}{l} E_r\ 2\,\pi\,r\,l = \dfrac{1}{\varepsilon_0}\,\rho\,\pi\,r^2\,l \quad \text{for} \quad \mathrm{r} < \mathrm{a} \\ E_r\ 2\,\pi\,r\,l = \dfrac{1}{\varepsilon_0}\,\rho\,\pi\,a^2\,l \quad \text{for} \quad a < \mathrm{r} < \mathrm{b} \end{array} \right.$$

=> $$\left| \begin{array}{l} E_r = \dfrac{\lambda(z)}{2\,\pi\,\varepsilon_0}\,\dfrac{r}{a^2} \quad \text{for} \quad \mathrm{r} < \mathrm{a} \\ E_r = \dfrac{\lambda(z)}{2\,\pi\,\varepsilon_0}\,\dfrac{1}{r} \quad \text{for} \quad a < \mathrm{r} < \mathrm{b} \end{array} \right.$$

Generalization

$$\lambda_0 \rightarrow \lambda(z)$$

=> **The (radial) Lorentz force on a particle of charge e inside the uniform cylinder is**

$$F_r = \frac{e}{\gamma^2}\,E_r = \frac{e}{2\,\pi\,\varepsilon_0\,\gamma^2}\,\lambda(z)\,\frac{r}{a^2}$$

1D TRANSVERSE ENVELOPE EQUATION (1/8)

- **Consider a particle in an ensemble of particles which obeys the single-particle equations**

$$\boxed{x' = p_x} \qquad \boxed{p'_x = \frac{F_x(x,s)}{\beta^2 \, E_{total}}} \qquad x' = \frac{d\,x}{d\,s}$$

- **The total force is** $F_x(x,s) = F_x^{ext} + F_x^{SC}$

- **Let's consider a particle distribution** $f(x, p_x, s)$**. Averaging over the particle distribution, we obtain the equations of motion for the centre of the beam**

$$< x >' = < p_x > \qquad < p_x >' = \frac{< F_x(x,s) >}{\beta^2 \, E_{total}} = \frac{< F_x^{ext} >}{\beta^2 \, E_{total}}$$

as $\boxed{< F_x^{SC} > = 0}$**, because of Newton's 3rd law**

1D TRANSVERSE ENVELOPE EQUATION (2/8)

- **For a linear machine, one has** $$\frac{F_x^{ext}}{\beta^2 \, E_{total}} = - K_x(s)\, x$$

=> $$< x >'' + K_x(s) < x > = 0$$

- **The 2nd moments satisfy the equations**

$$< x^2 >' = 2 < x \, x' > = 2 < x \, p_x >$$

$$< x \, p_x >' = < x' \, p_x > + < x \, p_x' > = < p_x^2 > - K_x(s) < x^2 > + < x \frac{F_x^{SC}}{\beta^2 \, E_{total}} >$$

$$< p_x^2 >' = 2 < p_x \, p_x' > = - 2 \, K_x(s) < x \, p_x > + 2 < p_x \frac{F_x^{SC}}{\beta^2 \, E_{total}} >$$

1D TRANSVERSE ENVELOPE EQUATION (3/8)

- **To study space-charge effects, we are interested in the position and momentum offsets of the particles from their respective averages, i.e.**

$$\Delta x = x - < x > \qquad \Delta p_x = p_x - < p_x >$$

=> $$< \Delta x^2 >' = 2 < \Delta x \, \Delta p_x >$$

$$< \Delta x \, \Delta p_x >' = < \Delta p_x^2 > - K_x (s) < \Delta x^2 > + < \Delta x \, \frac{F_x^{SC}}{\beta^2 \, E_{total}} >$$

$$< \Delta p_x^2 >' = - 2 \, K_x (s) < \Delta x \, \Delta p_x > + 2 < \Delta p_x \, \frac{F_x^{SC}}{\beta^2 \, E_{total}} >$$

- **Define the rms beam emittance** $\boxed{\varepsilon_{x,rms} = \sqrt{< \Delta x^2 > < \Delta p_x^2 > - < \Delta x \, \Delta p_x >^2}}$

and rms beam size $\boxed{\sigma_x = \sqrt{< \Delta x^2 >}}$

1D TRANSVERSE ENVELOPE EQUATION (4/8)

=> $$< \Delta p_x^2 > = \frac{\varepsilon_{x,rms}^2 + < \Delta x \, \Delta p_x >^2}{< \Delta x^2 >}$$

$$\sigma_x' = \frac{< \Delta x \, \Delta p_x >}{\sqrt{< \Delta x^2 >}}$$

$$\sigma_x'' = \frac{< \Delta x \, \Delta p_x >'}{\sqrt{< \Delta x^2 >}} - \frac{< \Delta x \, \Delta p_x >^2}{< \Delta x^2 >^{3/2}}$$

- **Finally, the 1D transverse (horiz.) envelope equation can be obtained**

$$\boxed{\sigma_x'' + K_x(s)\, \sigma_x - \frac{\varepsilon_{x,rms}^2}{\sigma_x^3} - \frac{< \Delta x \, F_x^{SC} >}{\sigma_x \, \beta^2 \, E_{total}} = 0}$$

1D TRANSVERSE ENVELOPE EQUATION (5/8)

- **Inserting the SC force, this yields**

$$\frac{F_x^{SC}}{\beta^2 E_{total}} = \frac{e\,\lambda}{\beta^2 E_{total}\, 2\,\pi\,\varepsilon_0\,\gamma^2}\,\frac{\Delta x}{a^2}$$

$$a = 2\,\sigma_x$$

$$\lambda = \frac{N_b\,e}{l} = N_l\,e$$

=> $$\frac{F_x^{SC}}{\beta^2 E_{total}} = K_{sc}\,\frac{\Delta x}{a^2}$$

with $$\boxed{K_{sc} = \frac{2\,N_l\,r_p}{\beta^2\,\gamma^3}}$$

$$r_p = \frac{e^2}{4\,\pi\,\varepsilon_0\,m_0\,c^2}$$

Therefore, $$\boxed{\frac{<\Delta x\,F_x^{SC}>}{\beta^2 E_{total}} = \frac{K_{sc}}{4}}$$

1D TRANSVERSE ENVELOPE EQUATION (6/8)

- The 1D envelope equation can finally be written

$$\boxed{a'' + K_x(s)\,a - \frac{\varepsilon_x^2}{a^3} - \frac{K_{sc}}{a} = 0}$$

$$\boxed{a = 2\sigma_x} \qquad \boxed{\varepsilon_x = 4\,\varepsilon_{x,rms}}$$

- Effect of space charge on the equilibrium beam size a_0, in the smooth approximation

$$K_x = \left(Q_{x0} / R\right)^2$$

1D TRANSVERSE ENVELOPE EQUATION (7/8)

=> The equilibrium beam size is therefore found from

$$\left(\frac{Q_{x0}}{R} \right)^2 a_0 - \frac{K_{sc}}{a_0} - \frac{\varepsilon_x^2}{a_0^3} = 0$$

which yields

$$\boxed{a_0^2 = \frac{\varepsilon_x R}{Q_{x0}} \left(\kappa + \sqrt{1 + \kappa^2} \right)} \qquad \boxed{\kappa = \frac{K_{sc} R}{2 \varepsilon_x Q_{x0}}}$$

- The beam size is significantly perturbed by the space-charge force when $\kappa \geq 1$
- If the beam size becomes larger than the vacuum chamber aperture, there will be a beam loss

1D TRANSVERSE ENVELOPE EQUATION (8/8)

- **For weak beam intensities, i.e.** $\kappa << 1$

$$a_0^2 = a_{00}^2 + \Delta a_{00}^2$$

with $a_{00}^2 = \dfrac{\varepsilon_x R}{Q_{x0}}$ $\Delta a_{00}^2 = \kappa \, a_{00}^2$

- **The parameter** a_{00} **describes the beam size in the absence of space charge. Interpreting** Δa_{00} **as a perturbation on the single-particle tune according to** $a_0^2 = \varepsilon_x R \,/ \left(Q_{x0} + \Delta Q_{x,\,\text{linear shift}}^{SC} \right)$ **, gives an expression for the shift of the single-particle tune due to space charge**

$$\boxed{\Delta Q_{x,\,\text{linear shift}}^{SC} = - \frac{K_{sc} R}{2 \varepsilon_x}}$$

2D TRANSVERSE ENVELOPE EQUATION (1/11)

- **Let's come back to the general case, i.e. consider a beam with unequal transverse beam sizes => The envelope equations are given by (a^2 must be replaced by $a\left(a+b\right)/2$ in horiz. SC term)**

$$a'' + K_x\, a - \frac{2\,K_{sc}}{a+b} - \frac{\varepsilon_x^2}{a^3} = 0$$

$$b'' + K_y\, b - \frac{2\,K_{sc}}{a+b} - \frac{\varepsilon_y^2}{b^3} = 0$$

=> Both transverse planes have thus to be treated jointly for high-intensity beams due to space-charge coupling

$$a = 2\sigma_x \qquad \varepsilon_x = 4\,\varepsilon_{x,rms}$$

$$b = 2\sigma_y \qquad \varepsilon_y = 4\,\varepsilon_{y,rms}$$

$$\Delta Q_{x,\,\text{linear shift}}^{SC} = -\frac{K_{sc}\,R^2}{Q_{x0}\,a_0\left(a_0+b_0\right)}$$

2D TRANSVERSE ENVELOPE EQUATION (2/11)

- The beam may execute some collective motion on top of equilibrium beam sizes a_0 and b_0

- Let the horizontal and vertical beam sizes be $\boxed{a(s) = a_0 - \Delta a(s)}$, $\boxed{b(s) = b_0 + \Delta b(s)}$

  where the perturbations Δa and Δb are considered small with respect to the equilibrium sizes

- Linearizing yields

$$\boxed{\Delta a'' + K_a \, \Delta a = K \, \Delta b}$$

$$\boxed{\Delta b'' + K_b \, \Delta b = K \, \Delta a}$$

$$K = \frac{2 K_{sc}}{\left(a_0 + b_0\right)^2}$$

$$K_a = 4 K_x - \frac{2 K_{sc} \left(2 a_0 + 3 b_0\right)}{a_0 \left(a_0 + b_0\right)^2}$$

$$K_b = 4 K_y - \frac{2 K_{sc} \left(2 b_0 + 3 a_0\right)}{b_0 \left(a_0 + b_0\right)^2}$$

=> The transverse beam sizes execute coupled oscillations

2D TRANSVERSE ENVELOPE EQUATION (3/11)

- **The equilibrium beam sizes a_0 and b_0 are found from the following equations**

$$K_x\, a_0 - \frac{2\,K_{sc}}{a_0 + b_0} - \frac{\varepsilon_x^2}{a_0^3} = 0$$

$$K_y\, b_0 - \frac{2\,K_{sc}}{a_0 + b_0} - \frac{\varepsilon_y^2}{b_0^3} = 0$$

- **Using the smooth approximation** $K_x = \left(Q_{x0} / R \right)^2$ $K_a = \left(Q_a / R \right)^2$ $K_y = \left(Q_{y0} / R \right)^2$ $K_b = \left(Q_b / R \right)^2$

and assuming small tune shifts, yields

2D TRANSVERSE ENVELOPE EQUATION (4/11)

$$Q_a = 2\,Q_{x0} + \Delta Q_a = 2\,Q_{x0} - \frac{K_{sc}\,R^2\left(2\,a_0 + 3\,b_0\right)}{2\,Q_{x0}\,a_0\left(a_0 + b_0\right)^2}$$

$$Q_b = 2\,Q_{y0} + \Delta Q_b = 2\,Q_{y0} - \frac{K_{sc}\,R^2\left(2\,b_0 + 3\,a_0\right)}{2\,Q_{y0}\,b_0\left(a_0 + b_0\right)^2}$$

- **The coupled equations can be re-written**

$$\frac{d^2\Delta\,a}{d\phi^2} + Q_a^2\,\Delta\,a = K\,R^2\,\Delta\,b$$

$$\frac{d^2\Delta\,b}{d\phi^2} + Q_b^2\,\Delta\,b = K\,R^2\,\Delta\,a$$

$$\phi = \Omega_0\,t$$

2D TRANSVERSE ENVELOPE EQUATION (5/11)

- **Far from the coupling resonance $Q_a = Q_b$, the solutions of the homogeneous equations (of the coupled oscillations) are given by**

$$\Delta a = \Delta a_0 \, e^{j Q_a \phi} \qquad \Delta b = \Delta b_0 \, e^{j Q_b \phi}$$

- **The formula we are looking for is the one of the previous page, giving Q_a …**

2D TRANSVERSE ENVELOPE EQUATION (6/11)

$$Q_a = 2Q_{x0} + \Delta Q_a = 2Q_{x0} - \frac{K_{sc}\, R^2 \left(2a_0 + 3b_0 \right)}{2Q_{x0}\, a_0 \left(a_0 + b_0 \right)^2}$$

$$\Delta Q^{SC}_{x,\,\text{linear shift}} = - \frac{K_{sc}\, R^2}{Q_{x0}\, a_0 \left(a_0 + b_0 \right)}$$

=> $$Q_a - 2Q_{x0} = \frac{\Delta Q^{SC}_{x,\,\text{linear shift}}}{2} \frac{\left(2a_0 + 3b_0 \right)}{\left(a_0 + b_0 \right)}$$

=> $$Q_a - 2Q_{x0} = \frac{\Delta Q^{SC}_{x,\,\text{linear shift}}}{2} \left(3 - \frac{\sigma_{x0}}{\sigma_{x0} + \sigma_{y0}} \right)$$

2D TRANSVERSE ENVELOPE EQUATION (7/11)

$$\Delta Q^{SC}_{x,\,\text{spread}} = -\,\Delta Q^{SC}_{x,\,\text{linear shift}}$$

=>

$$\Delta Q^{SC}_{x,\,\text{spread}} = \frac{2\,Q_{x0} - Q_a}{\frac{1}{2}\left(3 - \frac{\sigma_{x0}}{\sigma_{x0} + \sigma_{y0}}\right)}$$

Noted Q_{2x} on 1st slide

Equilibrium horiz. rms beam size

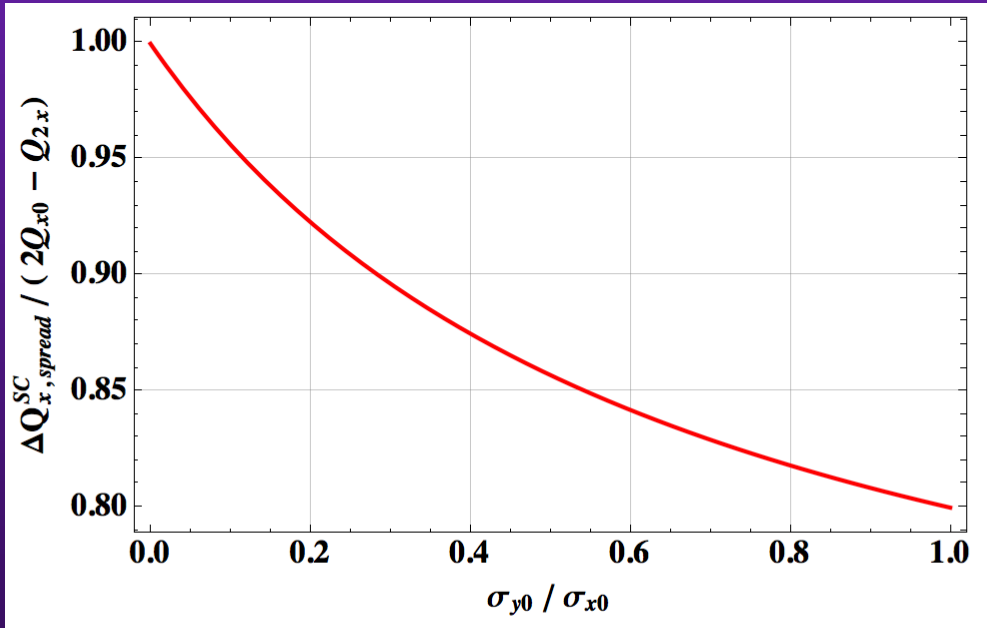
Equilibrium vertical rms beam size

2D TRANSVERSE ENVELOPE EQUATION (8/11)

- **This is the formula we were looking for! => It can also be written**

$$\Delta Q_{x,\,\mathrm{spread}}^{SC} = \frac{2\,Q_{x0} - Q_{2x}}{\frac{1}{2}\left(3 - \frac{1}{1 + \frac{\sigma_{y0}}{\sigma_{x0}}}\right)}$$

Only the ratio between the 2 transverse equilibrium beam sizes is needed

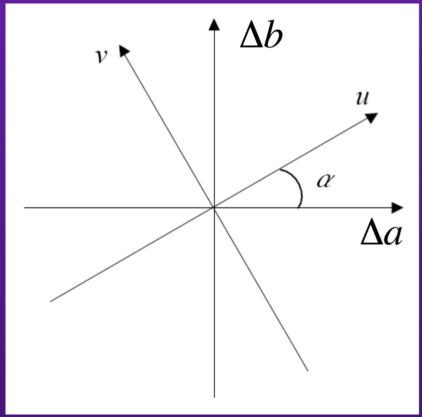


2D TRANSVERSE ENVELOPE EQUATION (9/11)

- **Close to the coupling resonance $Q_a = Q_b$, the solutions of the equations (of the coupled oscillations) are a bit more involved (see USPAS course) => The coupled oscillations can be solved by searching the normal (i.e. decoupled) modes (*u*,*v*) linked by a simple rotation**

$$\begin{pmatrix} \Delta a \\ \Delta b \end{pmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

- **The equations of the 2 normal modes can be found**



$$\frac{d^2 u}{d\phi^2} + Q_u^2\, u = 0 \qquad \frac{d^2 v}{d\phi^2} + Q_v^2\, v = 0$$

2D TRANSVERSE ENVELOPE EQUATION (10/11)

with (assuming small tune shifts)

$$Q_u = Q_a - \frac{|C|}{2}\tan\alpha \qquad Q_v = Q_b + \frac{|C|}{2}\tan\alpha$$

$$\tan(2\alpha) = \frac{|C|}{\Delta} \qquad |C| = \frac{R^2 K}{Q_0} \qquad \Delta = Q_b - Q_a \qquad Q_{x0} \approx Q_{y0} \approx Q_0$$

- $$\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha}$$ **=>** $$\tan\alpha = \frac{1}{|C|}\left(-\Delta \mp \sqrt{\Delta^2 + C^2}\right)$$

=> $$\frac{|C|}{2}\tan\alpha = \frac{1}{2}\left(-\Delta \mp \sqrt{\Delta^2 + C^2}\right)$$

2D TRANSVERSE ENVELOPE EQUATION (11/11)

=>

$$Q_u = Q_a - \frac{1}{2}\left(-\Delta \mp \sqrt{\Delta^2 + C^2}\right)$$

$$Q_v = Q_b + \frac{1}{2}\left(-\Delta \mp \sqrt{\Delta^2 + C^2}\right)$$

± depends on the sign of Δ => Should be the same sign as Δ

- When $|\Delta| >> |C|$, one recovers $Q_u \approx Q_a$ $Q_v \approx Q_b$