MEASUREMENT OF THE SPACE CHARGE TUNE SPREAD WITH A QUADRUPOLAR PICK-UP: NEW (GENERAL) FORMULA VS. "USUAL" ONE

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- **Follow-up of the talk given at the Space Charge meeting held on 18/12/2014(http://emetral.web.cern.ch/emetral/SomeOtherInterestingTalksAndPapers/SCtunespreadFromQPU_EM_18-12-14_v2.pdf), devoted to the re-derivation of the "usual" formula and some discussion about the more general one**

=> More discussion today about the more general formula vs. "usual" one

REMINDER ON "USUAL" FORMULA (1/2)

- Far from the coupling resonance $Q_a = Q_b$, the solutions of the homogeneous equations (of the coupled oscillations) are given by

$$\Delta a = \Delta a_0 \, e^{j Q_a \phi} \qquad \Delta b = \Delta b_0 \, e^{j Q_b \phi}$$

=>

$$\Delta Q_{x,\,\mathrm{spread}}^{SC\,,\,usual} = \frac{2 Q_{x0} - Q_a}{\frac{1}{2}\left(3 - \frac{\sigma_{x0}}{\sigma_{x0} + \sigma_{y0}}\right)}$$

Also noted Q_{2x}

Equilibrium horiz. rms beam size

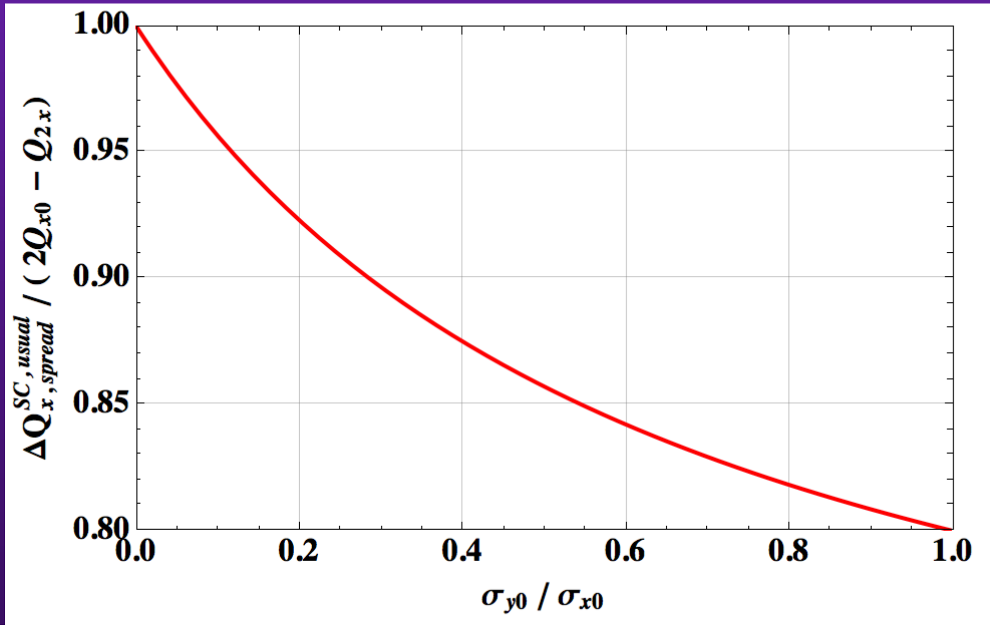
Equilibrium vertical rms beam size

REMINDER ON "USUAL" FORMULA (2/2)

- **It can also be written**

$$\Delta Q_{x,\,\mathrm{spread}}^{SC\,,\,usual} = \frac{2\,Q_{x0} - Q_{2x}}{\frac{1}{2}\left(3 - \frac{1}{1 + \frac{\sigma_{y0}}{\sigma_{x0}}}\right)}$$

Only the ratio between the 2 transverse equilibrium beam sizes is needed



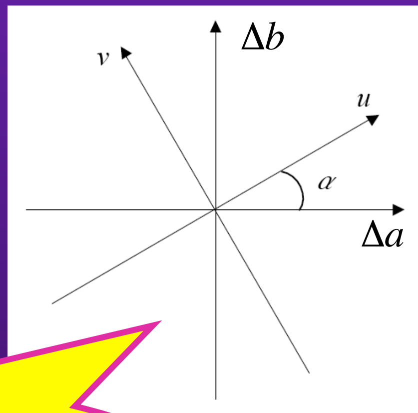
REMINDER ON MORE GENERAL FORMULA (1/3)

- **Close to the coupling resonance $Q_a = Q_b$, the solutions of the equations (of the coupled oscillations) are a bit more involved => The coupled oscillations can be solved by searching the normal (i.e. decoupled) modes (*u*,*v*) linked by a simple rotation**

$$\begin{pmatrix} \Delta a \\ \Delta b \end{pmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

- **The equations of the 2 normal modes can be found**

$$\frac{d^2 u}{d\phi^2} + Q_u^2 \, u = 0 \qquad \frac{d^2 v}{d\phi^2} + Q_v^2 \, v = 0$$



α = coupling angle:
1) = 0 for no coupling
2) = ± 45 deg for full coupling

REMINDER ON MORE GENERAL FORMULA (2/3)

with (assuming small tune shifts)

$$Q_u = Q_a - \frac{|C|}{2} \tan\alpha \qquad Q_v = Q_b + \frac{|C|}{2} \tan\alpha$$

$$\tan(2\alpha) = \frac{|C|}{\Delta} \qquad |C| = \frac{R^2 K}{Q_0} \qquad \Delta = Q_b - Q_a \qquad Q_{x0} \approx Q_{y0} \approx Q_0$$

- $$\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha}$$ => $$\tan\alpha = \frac{1}{|C|}\left(-\Delta \mp \sqrt{\Delta^2 + C^2}\right)$$

=> $$\frac{|C|}{2}\tan\alpha = \frac{1}{2}\left(-\Delta \mp \sqrt{\Delta^2 + C^2}\right)$$

REMINDER ON MORE GENERAL FORMULA (3/3)

=>

$$Q_u = Q_a - \frac{1}{2}\left(-\Delta \mp \sqrt{\Delta^2 + C^2} \right)$$

$$Q_v = Q_b + \frac{1}{2}\left(-\Delta \mp \sqrt{\Delta^2 + C^2} \right)$$

± depends on the sign of Δ => Should be the same sign as Δ

NEW FORMULA (1/5)

- **The new more general formula is thus given by**

$$\Delta Q_{x,\,\text{spread}}^{SC,new} = \frac{q}{\frac{1}{2}\left(3 - \frac{1}{1+x}\right)} - \frac{-\Delta \mp \sqrt{\Delta^2 + C^2}}{3 - \frac{1}{1+x}}$$

"Usual" formula

with

$$q = 2\,Q_{x0} - Q_u$$

This is the observable with the QPU

$$x = \frac{\sigma_{y0}}{\sigma_{x0}}$$

$$y = Q_{y0} - Q_{x0}$$

$$\Delta = 2\,y + \frac{3\,\Delta Q_{x,\,\text{spread}}^{SC,new}}{2}\left(1 - \frac{1}{x}\right)$$

$$|C| = \frac{2\,\Delta Q_{x,\,\text{spread}}^{SC,new}}{1+x}$$

NEW FORMULA (2/5)

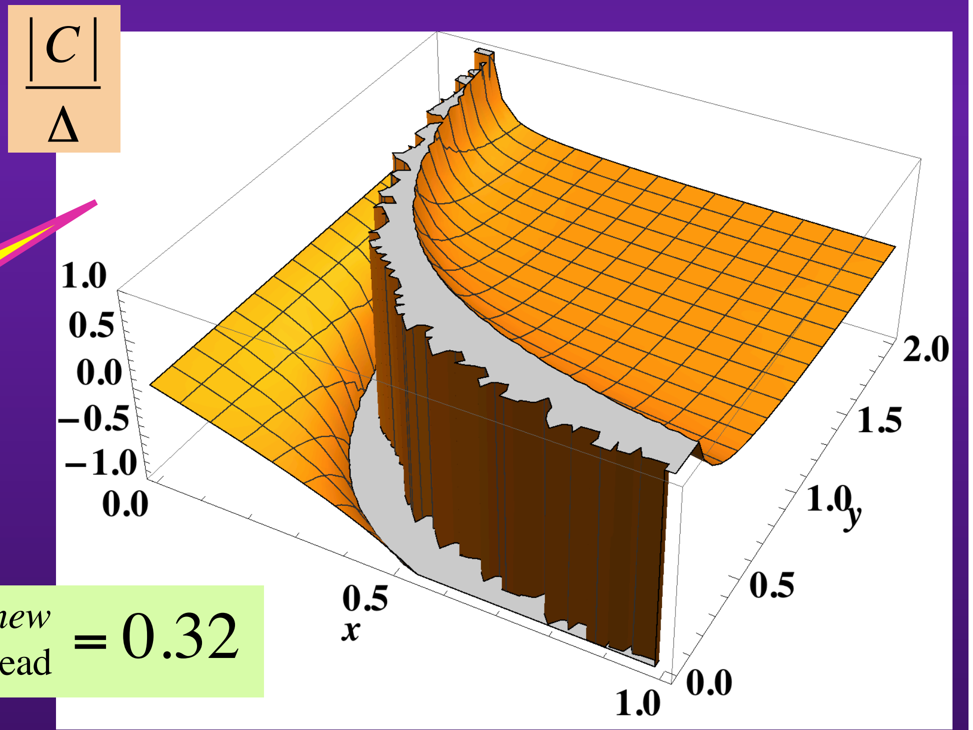
- **In the general case there are thus 2 contributions in the quadrupolar tune shift**
  - **Space charge tune shift (spread)**
  - **Coupling between the 2 transverse planes due to space charge**

- **The "usual" formula is obtained when the space charge coupling is negligible, i.e.**

$$\left|\frac{C}{\Delta}\right| = \frac{1}{\left|\frac{y}{\Delta Q_{x,\,\mathrm{spread}}^{SC,new}}\left(1+x\right)+\frac{3}{4}\left(x-\frac{1}{x}\right)\right|} << 1$$

$$\Delta > 0 \Leftrightarrow$$

$$y > \frac{3\,\Delta Q_{x,\,\mathrm{spread}}^{SC,new}}{4}\left(\frac{1}{x}-1\right)$$



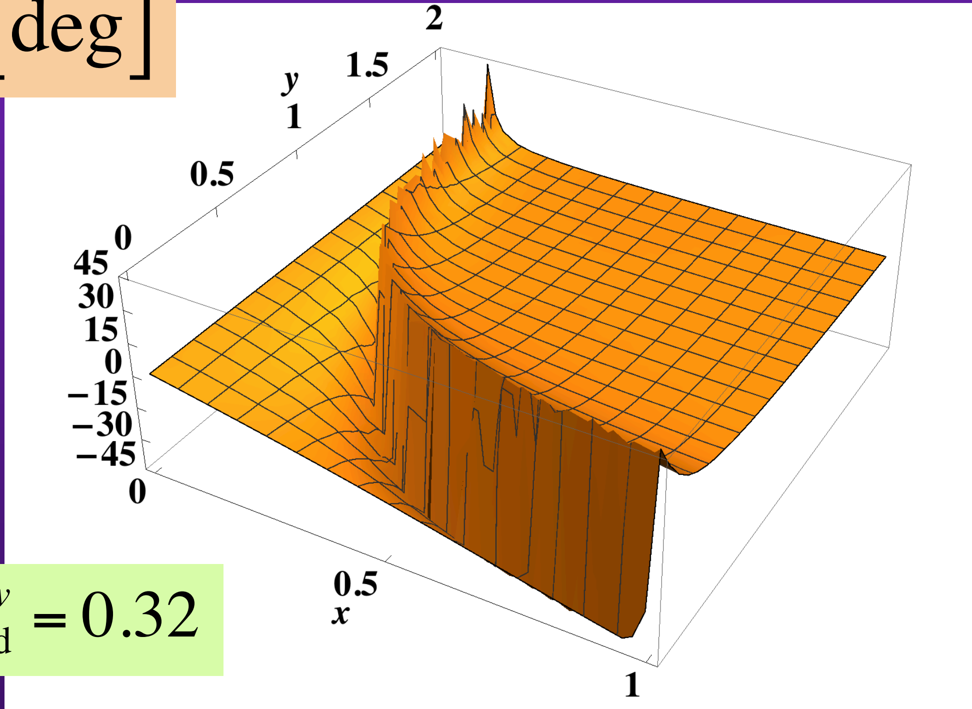
$$\Delta Q_{x,\,\mathrm{spread}}^{SC\,,\,new} = 0.32$$

NEW FORMULA (3/5)

- The coupling angle is given by

$$\alpha = \frac{1}{2} \text{ArcTan}\left[\frac{1}{\frac{y}{\Delta Q^{SC,new}_{x,\,\text{spread}}}\left(1+x\right) + \frac{3}{4}\left(x - \frac{1}{x}\right)} \right]$$

$\alpha\,[\text{deg}]$



$\Delta Q^{SC,new}_{x,\,\text{spread}} = 0.32$

NEW FORMULA (4/5)

- **Solving the equation of page 1/5, yields**

$$\Delta Q_{x,\text{spread}}^{SC,new}(x,y,q) = \frac{1}{6+9x+6x^2}\left[q\left(3+7x+7x^2+3x^3\right)+4xy+10x^2y+6x^3y \mp (1+x)\sqrt{q^2\left(9-2x^2+9x^4\right)+4qx\left(-6-x+6x^2+9x^3\right)y+4x^2(2+3x)^2y^2}\right]$$

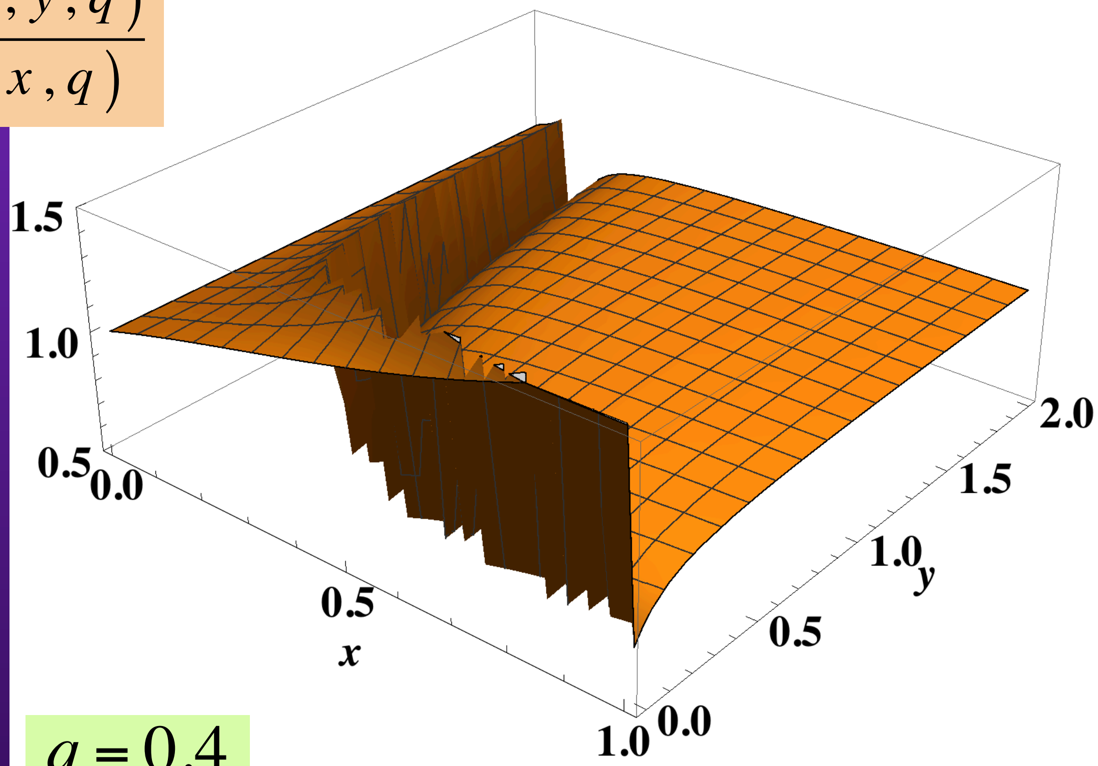
$-$ if $\Delta > 0$

$+$ if $\Delta < 0$

$\Delta > 0 \Leftrightarrow$

$$y > \frac{3\,\Delta Q_{x,\text{spread}}^{SC,new}}{4}\left(\frac{1}{x}-1\right)$$

$$\frac{\Delta Q_{x,\text{spread}}^{SC,new}(x,y,q)}{\Delta Q_{x,\text{spread}}^{SC,usual}(x,q)}$$



$q = 0.4$

NEW FORMULA (5/5)

- **Some numerical applications**

$$\Delta Q^{SC\,,\,usual}_{x,\,\text{spread}}\left(1\,,0.4\right)=0.32$$

$$\Delta Q^{SC\,,\,new}_{x,\,\text{spread}}\left(1\,,2\,,0.4\right)=0.315$$

=> $$\frac{\Delta Q^{SC\,,\,new}_{x,\,\text{spread}}\left(1\,,2\,,0.4\right)}{\Delta Q^{SC\,,\,usual}_{x,\,\text{spread}}\left(1\,,0.4\right)}\approx 0.985$$

$$\Delta Q^{SC\,,\,usual}_{x,\,\text{spread}}\left(1\,,0.4\right)=0.32$$

$$\Delta Q^{SC\,,\,new}_{x,\,\text{spread}}\left(1\,,0.03\,,0.4\right)=0.244$$

=> $$\frac{\Delta Q^{SC\,,\,new}_{x,\,\text{spread}}\left(1\,,0.03\,,0.4\right)}{\Delta Q^{SC\,,\,usual}_{x,\,\text{spread}}\left(1\,,0.4\right)}\approx 0.76$$

$$\Delta Q^{SC\,,\,usual}_{x,\,\text{spread}}\left(0.1\,,0.4\right)=0.38$$

$$\Delta Q^{SC\,,\,new}_{x,\,\text{spread}}\left(0.1\,,0.03\,,0.4\right)=0.41$$

=> $$\frac{\Delta Q^{SC\,,\,new}_{x,\,\text{spread}}\left(0.1\,,0.03\,,0.4\right)}{\Delta Q^{SC\,,\,usual}_{x,\,\text{spread}}\left(0.1\,,0.4\right)}\approx 1.06$$

=> The "usual" formula should not work for the "usual" cases of machines running close to the coupling resonance with almost round beams…

To be checked…