TRANSVERSE BEAM DYNAMICS

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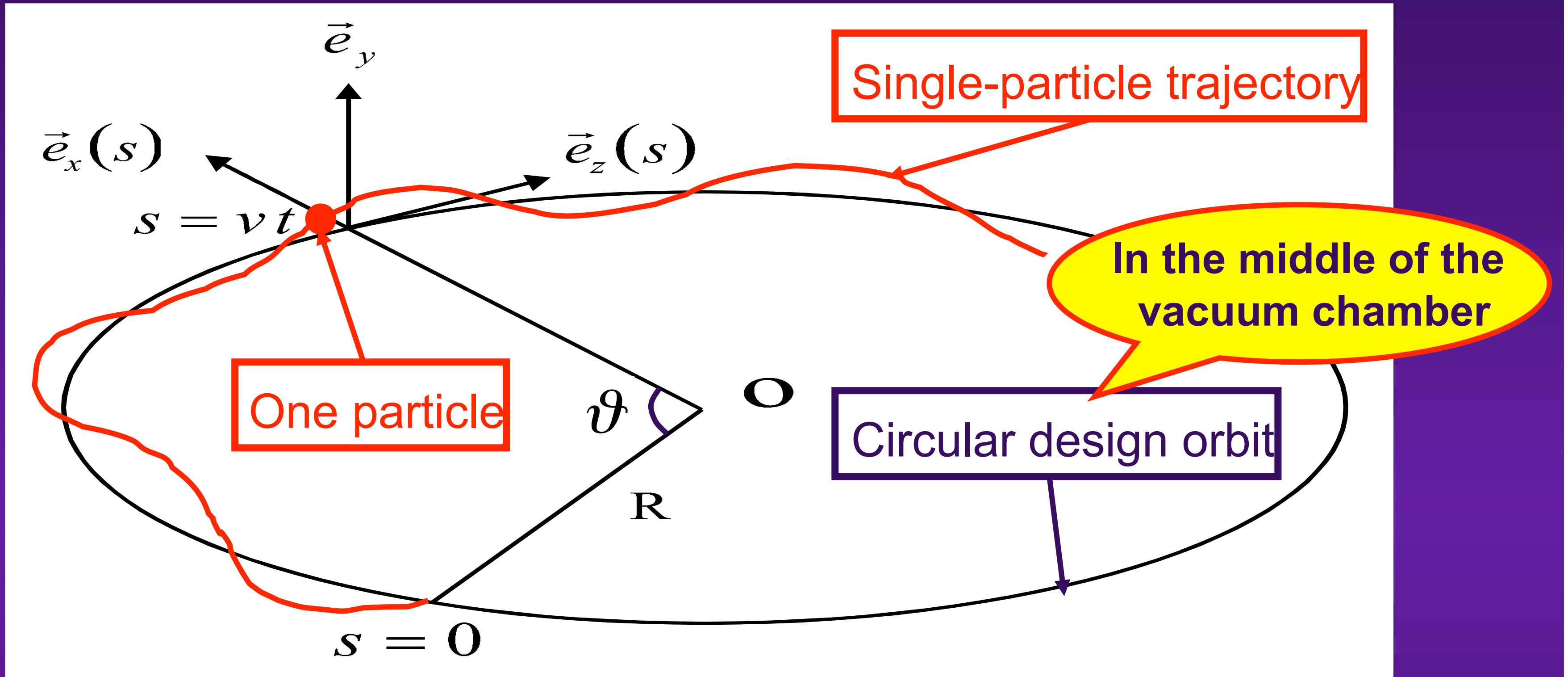
- **Introduction**
- **Multipole field expansion**
- **Equations of motion (general & Hill's equation and solution)**
- **Matrix formalism and Twiss parameters**
- **Thin lens approximation and FODO cell**
- **Beam emittance, envelope and divergence**
- **Dispersion**
- **Normalized Floquet's coordinates**
- **Chromaticity and its correction**
- **Nonlinearities and resonances**

INTRODUCTION (1/6)



- The motion of a charged particle (proton) in a beam transport channel or a circular accelerator is governed by the **LORENTZ FORCE**

$$\vec{F} = \frac{d\,\vec{p}}{d\,t} = e\left(\cancel{\vec{E}} + \vec{v} \times \vec{B}\right)$$

- The motion of particle beams under the influence of the Lorentz force is called **BEAM OPTICS**

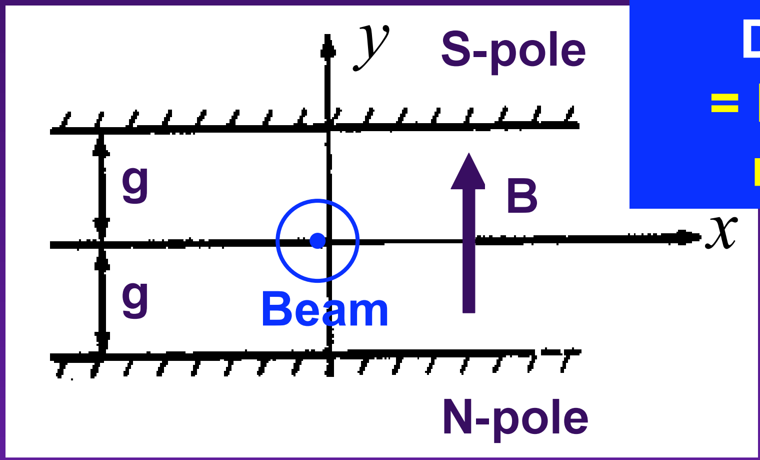
INTRODUCTION (2/6)

- **The Lorentz force is applied as a**
  - **BENDING FORCE (using DIPOLES) to guide the particles along a predefined ideal path, the DESIGN ORBIT, on which – ideally – all particles should move**
  - **FOCUSING FORCE (using QUADRUPOLES) to confine the particles in the vicinity of the ideal path, from which most particles will unavoidably deviate**

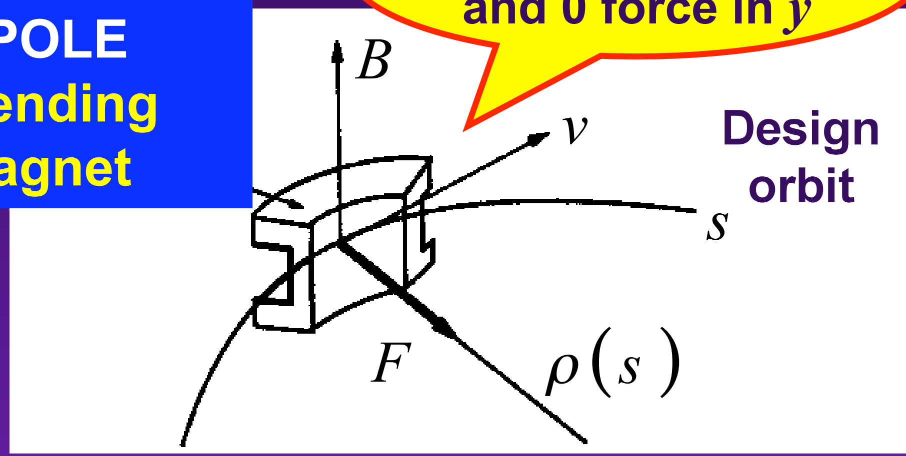
- **LATTICE = Arrangement of magnets along the design orbit**

- **The ACCELERATOR DESIGN is made considering the beam as a collection of non-interacting single particles**

INTRODUCTION (3/6)



DIPOLE = Bending magnet



Constant force in x and 0 force in y

⇒ **A particle, with a constant energy, describes a circle in equilibrium between the centripetal magnetic force and the centrifugal force**

$$e\, v\, B = m \frac{v^2}{\rho} \quad \Rightarrow \quad B\, \rho = \frac{p}{e}$$

- **BEAM RIGIDITY (in useful units)**

$$B\, \rho\, [\mathrm{Tm}] = 3.3356\; p_0\, [\mathrm{GeV/c}]$$

Magnetic field

Curvature radius of the dipoles

Beam momentum

INTRODUCTION (4/6)

- LEP vs LHC magnets (in same tunnel) => A change in technology

	LEP	LHC
ρ [m]	3096.175	2803.95
p_0 [GeV/c]	104	7000
B [T]	0.11	8.33

Room-temperature coils

Superconducting coils

- As the machine has to be closed $2\pi = N_d \, \vartheta_d$ $l_d = \rho \, \vartheta_d$

Total number of dipoles

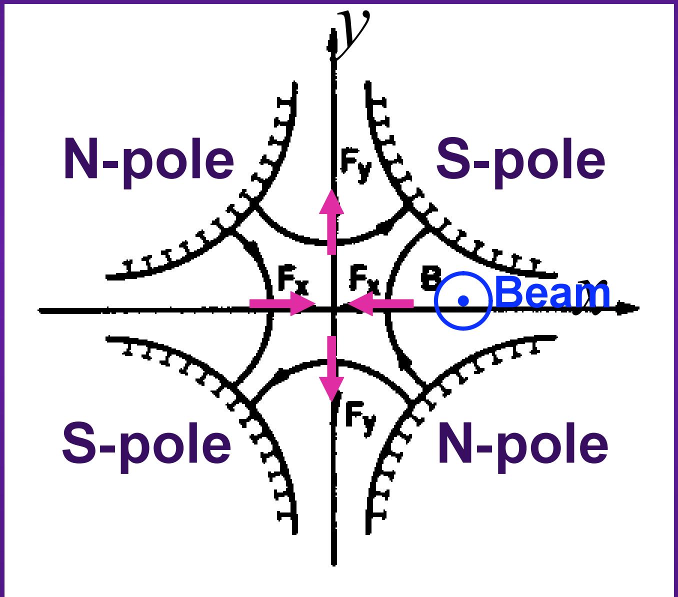
Bending angle of a dipole

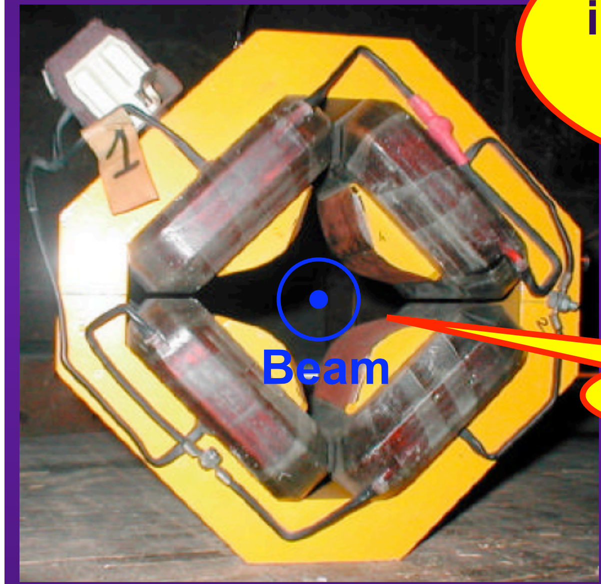
Length of a dipole

INTRODUCTION (5/6)

QUADRUPOLE = Focusing magnet

In x (and Defocusing in y) $\Rightarrow$ F-type. Permutating the N- and S- poles gives a D-type



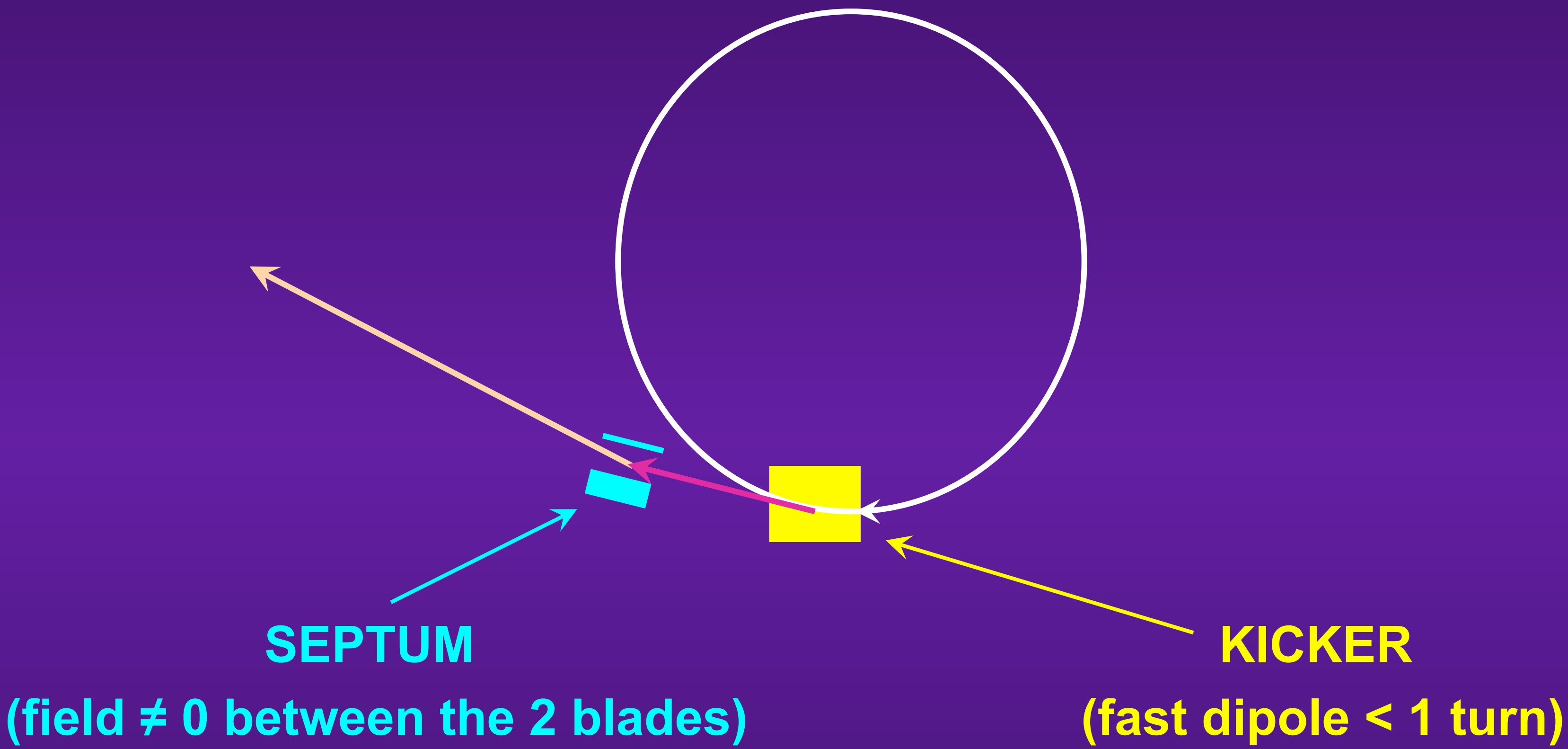


Linear force in x&y

… and SEXTUPOLES (6 poles), OCTUPOLES (8 poles), etc.

INTRODUCTION (6/6)

- (fast) EXTRACTION from a ring



- (fast) INJECTION into a ring => Reverse process

MULTIPOLE FIELD EXPANSION (1/7)

- In a vacuum environment in the vicinity of the design orbit, the following Maxwell equations hold

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = 0$$

- $\vec{\nabla} \times \vec{B} = 0$ => $\vec{B} = -\vec{\nabla} U$

With *U* a magnetic scalar potential

$$\Delta U \equiv \nabla^2 U = 0$$

Laplace equation

- Assumption that the field does not vary along the z-axis (as is the case for long magnets far from the ends) and that there are only transverse field components (no solenoid field)

MULTIPOLE FIELD EXPANSION (2/7)

=> $$U(x,y) = \sum_{n=1}^{\infty} U_n(x,y)$$

$$U_n(x,y) = -\frac{p}{e}\frac{A_n}{n!}(x + j\,y)^n$$

Decomposition of the magnetic potential as a sum of independent multipole terms

- The real and imaginary parts are 2 independent solutions of the Laplace equation and they differentiate between 2 classes of magnet orientation

MULTIPOLE FIELD EXPANSION (3/7)

- The imaginary part has the so-called mid-plane symmetry, i.e.

$$\mathrm{Im}\left[U_n(x,y)\right] = -\,\mathrm{Im}\left[U_n(x,-y)\right]$$

and the magnetic field components for the nth order multipoles are

$$B_{nx}(x,y) = -\frac{\partial}{\partial x}\mathrm{Im}\left[U_n(x,y)\right] = \frac{p}{e}A_n \sum_{m=0}^{(n-2)/2} (-1)^m \frac{x^{n-2m-2}}{(n-2m-2)!}\frac{y^{2m+1}}{(2m+1)!}$$

$$B_{ny}(x,y) = -\frac{\partial}{\partial y}\mathrm{Im}\left[U_n(x,y)\right] = \frac{p}{e}A_n \sum_{m=0}^{(n-1)/2} (-1)^m \frac{x^{n-2m-1}}{(n-2m-1)!}\frac{y^{2m}}{(2m)!}$$

MULTIPOLE FIELD EXPANSION (4/7)

- The mid-plane symmetry yields

$$B_{nx}\left(x,y\right)=-B_{nx}\left(x,-y\right)$$

$$B_{ny}\left(x,y\right)=B_{ny}\left(x,-y\right)$$

- There is no horizontal field in the mid-plane, $B_{nx}\left(x,0\right)=0$, and a particle traveling in the horizontal mid-plane will remain in this plane

- The magnets are called **UPRIGHT or REGULAR magnets**

- The coefficient A_n is called the (normalized) multipole strength parameter

$$\boxed{A_n=\frac{e}{p}\,\frac{\partial^{\,n-1}}{\partial x^{\,n-1}}B_{ny}=-\frac{e}{p}\left(-1\right)^{n/2}\frac{\partial^{\,n-1}}{\partial y^{\,n-1}}B_{nx}}$$

MULTIPOLE FIELD EXPANSION (5/7)

- **The magnets derived from the real solution of the potential are called ROTATED or SKEW magnets and the magnetic field components for the nth order skew multipoles are**

$$B_{nx}(x,y) = -\frac{\partial}{\partial x}\mathrm{Re}\left[U_n(x,y)\right]$$
$$= \frac{p}{e}\underline{A}_n \sum_{m=0}^{(n-1)/2} (-1)^m \frac{x^{n-2m-1}}{(n-2m-1)!}\frac{y^{2m}}{(2m)!}$$

$$B_{ny}(x,y) = -\frac{\partial}{\partial y}\mathrm{Re}\left[U_n(x,y)\right]$$
$$= \frac{p}{e}\underline{A}_n \sum_{m=0}^{n/2} (-1)^m \frac{x^{n-2m}}{(n-2m)!}\frac{y^{2m-1}}{(2m-1)!}$$

MULTIPOLE FIELD EXPANSION (6/7)

- **The coefficient $\underline{A}_n$ is called the (normalized) skew multipole strength parameter**

$$\boxed{\underline{A}_n = \frac{e}{p} \frac{\partial^{n-1}}{\partial x^{n-1}} B_{nx} = \frac{e}{p} \left(-1 \right)^{n/2} \frac{\partial^{n-1}}{\partial y^{n-1}} B_{ny}}$$

- **The skew magnets differ from the regular ones only by a rotation about the z-axis by an angle (where *n* is the order of the multipole)**

$$\boxed{\phi_n = \frac{\pi}{2\,n}}$$

MULTIPOLE FIELD EXPANSION (7/7)

- **Regular and skew multipole fields for the first 4 orders**

$$\boxed{A_2 = K}$$

$$\boxed{A_3 = S}$$

$$\boxed{A_4 = O}$$

$$\boxed{\underline{A}_2 = \underline{K}}$$

$$\boxed{\underline{A}_3 = \underline{S}}$$

$$\boxed{\underline{A}_4 = \underline{O}}$$

Regular multipole fields

Dipole	$\frac{e}{p} B_{1x} = 0$	$\frac{e}{p} B_{1y} = \frac{1}{\rho_x}$
Quadrupole	$\frac{e}{p} B_{2x} = K y$	$\frac{e}{p} B_{2y} = K x$
Sextupole	$\frac{e}{p} B_{3x} = S x y$	$\frac{e}{p} = B_{3y} = \frac{1}{2} S (x^2 - y^2)$
Octupole	$\frac{e}{p} B_{4x} = \frac{1}{6} O (3x^2 y - y^3)$	$\frac{e}{p} B_{4y} = \frac{1}{6} O (x^3 - 3 x y^2)$

Skew multipole fields

Dipole ($\phi = 90^\circ$)	$\frac{e}{p} B_{1x} = -\frac{1}{\rho_y}$	$\frac{e}{p} B_{1y} = 0$
Quadrupole ($\phi = 45^\circ$)	$\frac{e}{p} B_{2x} = \underline{K} x$	$\frac{e}{p} B_{2y} = -\underline{K} y$
Sextupole ($\phi = 30^\circ$)	$\frac{e}{p} B_{3x} = \frac{1}{2} \underline{S} (x^2 - y^2)$	$\frac{e}{p} B_{3y} = -\underline{S} x y$
Octupole ($\phi = 22.5^\circ$)	$\frac{e}{p} B_{4x} = \frac{1}{6} \underline{O} (x^3 - 3 x y^2)$	$\frac{e}{p} B_{4y} = -\frac{1}{6} \underline{O} (3x^2 y - y^3)$

EQUATIONS OF MOTION (1/9)

- General equations of motion for a charged particle in a magnetic field (in absence of electric field)

$$x'' - \frac{\sigma''}{\sigma'}\,x' = k_x\,h - \left(1+\delta\right)^{-1}\frac{e}{p_0}\,\sigma'\left(h\,B_y - y'\,B_t\right)$$

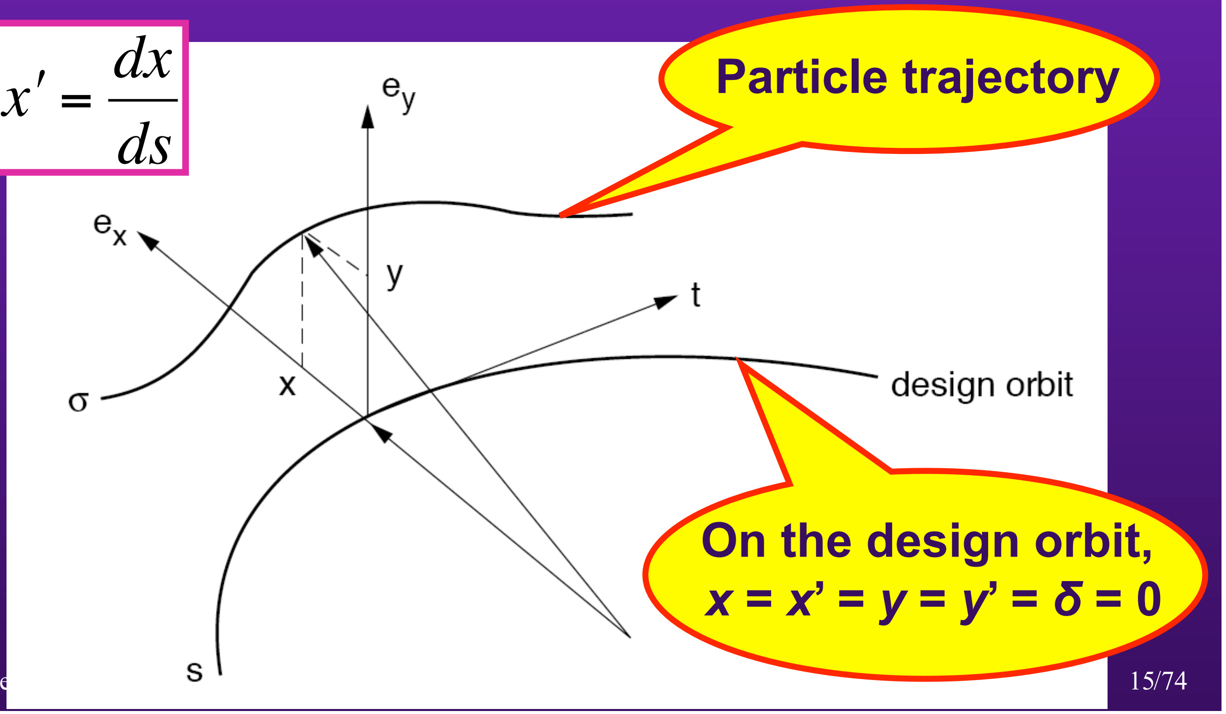
$$y'' - \frac{\sigma''}{\sigma'}\,y' = k_y\,h + \left(1+\delta\right)^{-1}\frac{e}{p_0}\,\sigma'\left(h\,B_x - x'\,B_t\right)$$

with $\boxed{\delta = \frac{p-p_0}{p_0} = \frac{\Delta p}{p_0}}$ $\boxed{x' = \frac{dx}{ds}}$

$$h = 1 + k_x\,x + k_y\,y$$

$$\sigma' = \sqrt{h^2 + x'^2 + y'^2}$$

$$\boxed{\rho_{x,y}\left(s\right) = \frac{1}{k_{x,y}\left(s\right)}}$$



EQUATIONS OF MOTION (2/9)

- **General equations of motion expanded to the 2nd order in *x*, *y* and *δ* (without tangential fields)**

$$x'' + \left(K_0 + \frac{1}{\rho_x^2} \right) x = \underline{K}_0 \, y + \frac{\delta}{\rho_x} - \frac{\delta^2}{\rho_x} + \left(K_0 + \frac{2}{\rho_x^2} \right) x \, \delta - \frac{1}{2} S_0 \left(x^2 - y^2 \right) - \underline{K}_0 \, y \, \delta + \underline{S}_0 \, x \, y$$

$$y'' - \left(K_0 - \frac{1}{\rho_y^2} \right) y = \underline{K}_0 \, x + \frac{\delta}{\rho_y} - \frac{\delta^2}{\rho_y} - \left(K_0 - \frac{2}{\rho_y^2} \right) y \, \delta + \frac{1}{2} \underline{S}_0 \left(x^2 - y^2 \right) - \underline{K}_0 \, x \, \delta + S_0 \, x \, y$$

  - **If we consider the on-momentum particle (*δ* = 0) and if we assume that the design orbit lies only in the horizontal plane, then the equations of motion simplifies to**

$$x'' + \left(K_0 + \frac{1}{\rho_x^2} \right) x = \underline{K}_0 \, y - \frac{1}{2} S_0 \left(x^2 - y^2 \right) + \underline{S}_0 \, x \, y$$

$$y'' - K_0 \, y = \underline{K}_0 \, x + \frac{1}{2} \underline{S}_0 \left(x^2 - y^2 \right) + S_0 \, x \, y$$

EQUATIONS OF MOTION (3/9)

- If we consider now the on-momentum particle (δ = 0), if we assume that the design orbit lies only in the horizontal plane, and if we consider only the presence of dipoles and normal (regular) quadrupoles, then the equations of motion simplifies to the so-called **HILL'S EQUATION**

Gradient focusing

Weak sector magnet focusing

u = *x* or *y*

$$x'' + \left(K_0 + \frac{1}{\rho_x^2} \right) x = 0 \qquad \Rightarrow \qquad u'' + K(s)\, u = 0$$

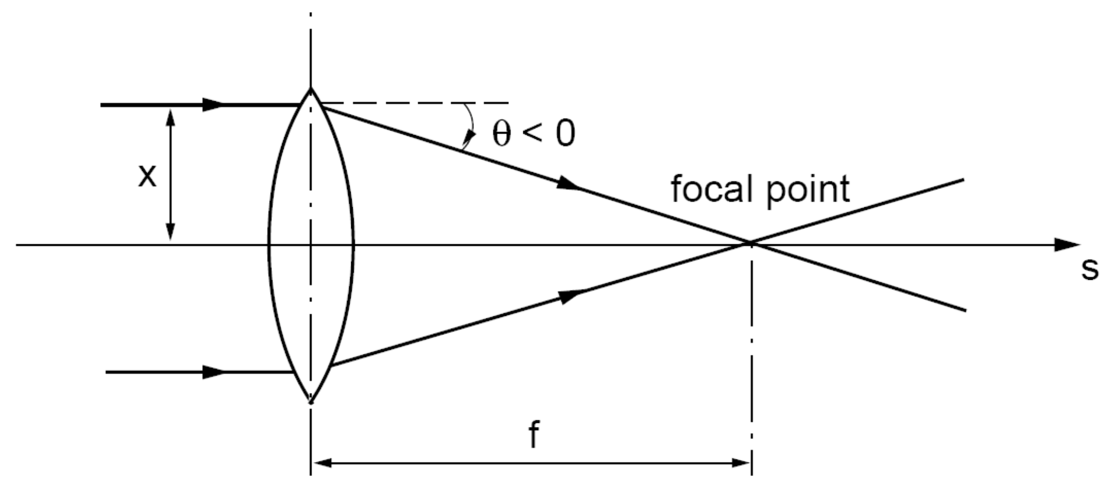
$$y'' - K_0\, y = 0$$

Periodic function of the s-coordinate with period $L = C / N_{cell}$, where C is the accelerator circumference and N_{cell} the number of "cells" repeated

EQUATIONS OF MOTION (4/9)

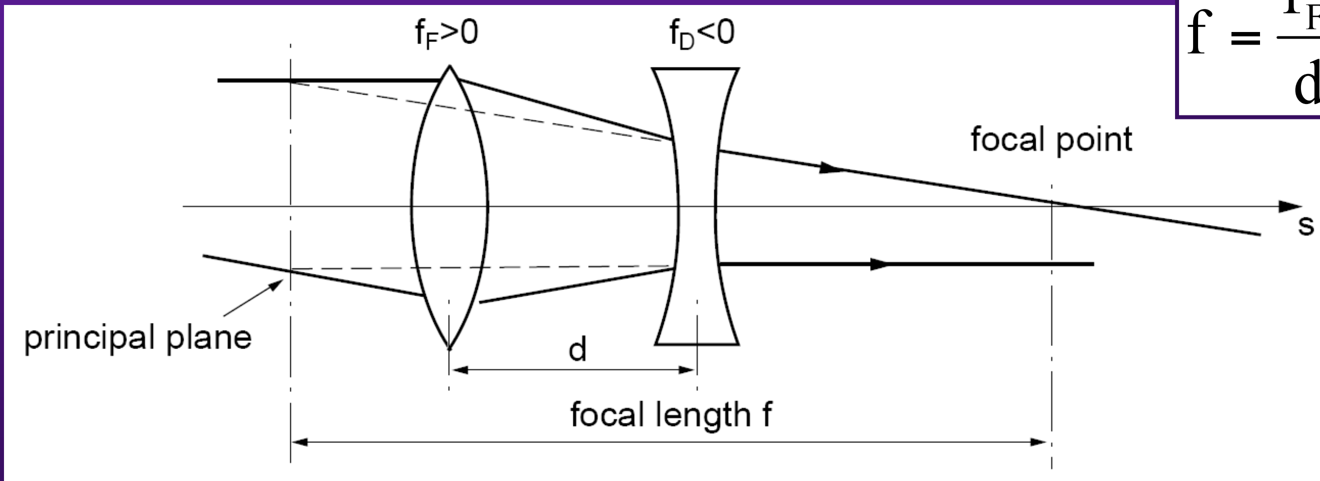
Reminder on light optics

Principle of focusing for light



$$\tan\theta = -\frac{x}{f}$$

Principle of STRONG FOCUSING for light



$$f = \frac{f_F^2}{d} > 0 \left(\text{when } f_D = -f_F \right)$$

=> Focusing in both planes

EQUATIONS OF MOTION (5/9)

Coming back to the case of a strong focusing lattice

$$\int_0^l u''\, ds = u'(l) - u'(0) = \tan\theta$$

$$\int_0^l K(s)\, u\, ds \approx K\, u\, l$$

=> $\tan\theta \approx -K\, u\, l$

By analogy with the expression of the focal length of a glass lens, we define the focal length of a quadrupole

$$\tan\theta = -\frac{u}{f}$$

=>

$$\frac{1}{f} = \pm K_0\, l$$

Assuming $K(s) = K_0 > 0$

+ in focusing plane and – in defocusing plane

EQUATIONS OF MOTION (6/9)

- The solution of the Hill's equation is a pseudo-harmonic oscillation with varying amplitude and frequency called **BETATRON OSCILLATION**

$$u(s) = \mathrm{a}\sqrt{\beta(s)}\,\cos\left[\mu(s) - \varphi\right]$$

Phase function

Betatron function

with

$$\mu(s) = \int_{s_0}^{s} \frac{dt}{\beta(t)}$$

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + K(s)\beta^2 = 1$$

- Phase advance per period or cell (of length *L*)

$$\mu = \int_{s}^{s+L} \frac{dt}{\beta(t)}$$

EQUATIONS OF MOTION (7/9)

- The oscillation's local frequency *f* (s) is related to the phase function by

$$\mu'(s) = 2\pi f(s)$$

- The number of betatron oscillations executed by particles traveling once around the machine circumference *C* is called the **BETATRON TUNE** and is given by

$$Q = \int_{s}^{s+C} f(t)\, dt$$

=> $$\boxed{Q = \frac{N_{cell}\,\mu}{2\pi} = \frac{1}{2\pi} \int_{s}^{s+C} \frac{dt}{\beta(t)}}$$

EQUATIONS OF MOTION (8/9)

- Case of a drift space (where $K = 0$)

$$\frac{1}{2}\beta\,\beta'' - \frac{1}{4}\beta'^2 + K(s)\,\beta^2 = 1 \quad \Rightarrow \quad \frac{1}{2}\beta\,\beta'' - \frac{1}{4}\beta'^2 = 1$$

=> The solution is (with $\beta(0) = \beta_0$)

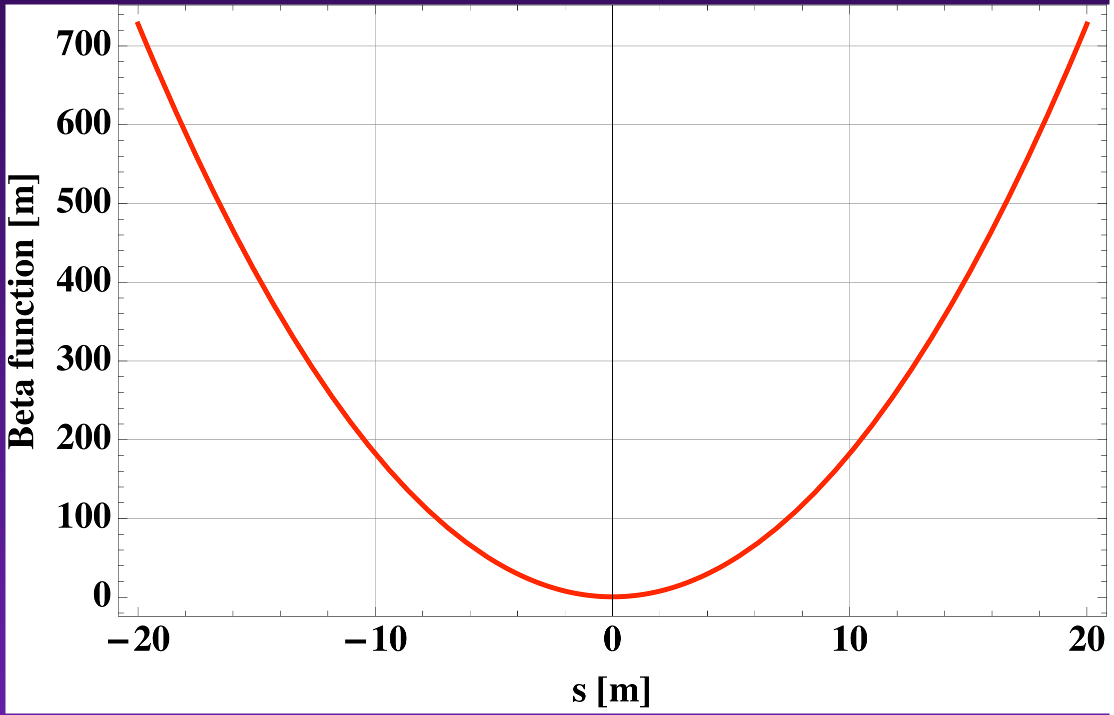
$$\beta(s) = \beta_0 + \beta_0'\,s + \frac{1}{\beta_0}\left[1 + \frac{\beta_0'^{\,2}}{4}\right] s^2$$

- When $\beta_0' = 0$, the betatron function reduces to

Case of a symmetric point, as a LHC IP

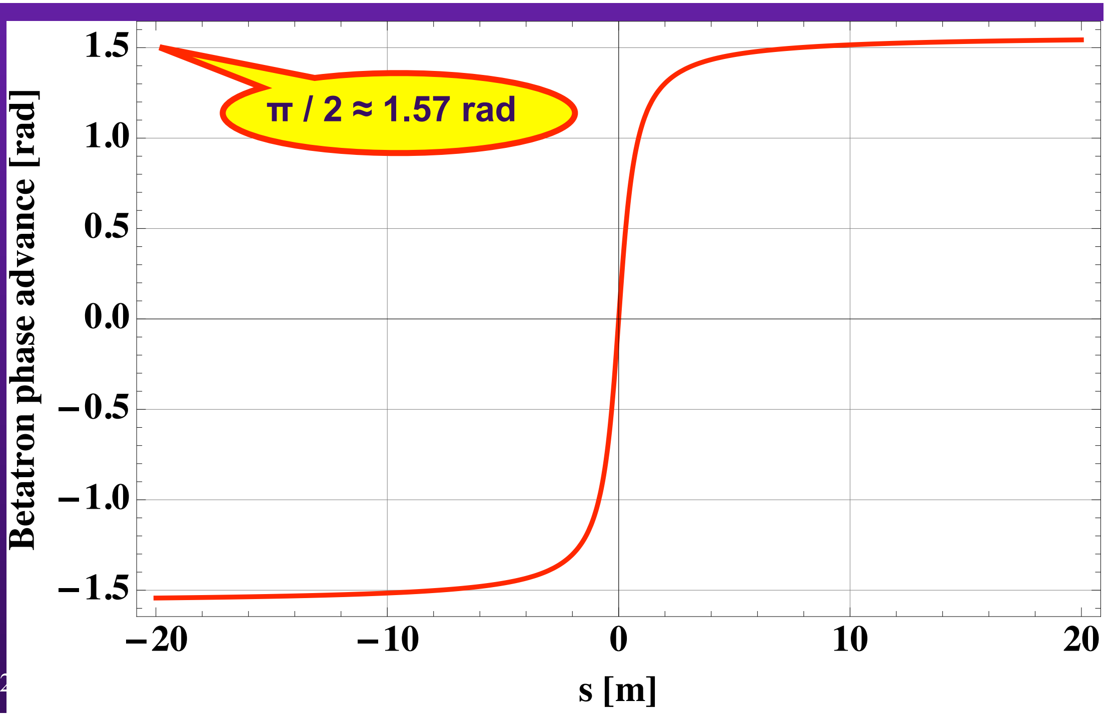
$$\boxed{\beta(s) = \beta_0\left[1 + \left(\frac{s}{\beta_0}\right)^2\right]}$$

EQUATIONS OF MOTION (9/9)



- The betatron phase advance is given by

$$\mu(s) = \int_0^s \frac{dt}{\beta(t)} = \arctan\left(\frac{s}{\beta^*}\right)$$



MATRIX FORMALISM AND TWISS PARAMETERS (1/5)

- **Using the matrix formalism, one can write**

$$\begin{bmatrix} u(s) \\ u'(s) \end{bmatrix} = \underbrace{M(s/s_0)}_{\text{Transfer matrix between } s_0 \text{ and } s} \begin{bmatrix} u(s_0) \\ u'(s_0) \end{bmatrix}$$

with

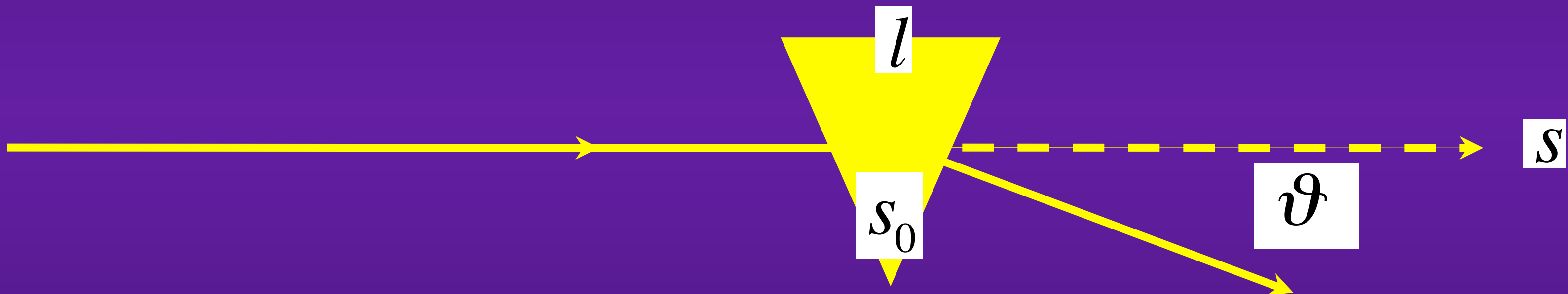
$$M(s/s_0) = \begin{bmatrix} \sqrt{\dfrac{\beta(s)}{\beta_0}}\left[\cos\Delta\mu(s) + \alpha_0 \sin\Delta\mu(s)\right] & \sqrt{\beta(s)\,\beta_0}\,\sin\Delta\mu(s) \\ \dfrac{\alpha_0 - \alpha(s)}{\sqrt{\beta(s)\,\beta_0}}\cos\Delta\mu(s) - \dfrac{1+\alpha(s)\,\alpha_0}{\sqrt{\beta(s)\,\beta_0}}\sin\Delta\mu(s) & \sqrt{\dfrac{\beta_0}{\beta(s)}}\left[\cos\Delta\mu(s) - \alpha(s)\sin\Delta\mu(s)\right] \end{bmatrix}$$

$$\boxed{\alpha(s) = -\frac{\beta'(s)}{2}} \qquad \Delta\mu(s) = \mu(s) - \mu(s_0)$$

MATRIX FORMALISM AND TWISS PARAMETERS (2/5)

- **Effect of the kick of a dipole => Consider a small dipole placed at s_0 yielding an angular kick**

$$\vartheta = \frac{l}{\rho} = \frac{e}{p} \Delta B \, l$$



The motion of a particle traveling on the design orbit for s > s_0 (before it is 0) is (from the previous general matrix formula)

$$x(s) = \sqrt{\beta(s)\,\beta_0}\, \sin\left[\mu(s) - \mu_0\right] \vartheta$$

$$\beta(s_0) = \beta_0$$

$$\mu(s_0) = \mu_0$$

MATRIX FORMALISM AND TWISS PARAMETERS (3/5)

- The transfer matrix over one period = TWISS MATRIX and is given by

$$M(s) \equiv M(s+L/s) = \begin{bmatrix} \cos\mu + \alpha(s)\sin\mu & \beta(s)\sin\mu \\ -\gamma(s)\sin\mu & \cos\mu - \alpha(s)\sin\mu \end{bmatrix}$$

with $$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

General stability criterion for a Twiss matrix => | Tr [M (s)] | ≤ 2

- Determinant of the Twiss matrix $|M(s)| = 1$

- Trace of the Twiss matrix $\mathrm{Tr}\left[M(s)\right] = 2\cos\mu$

$\beta(s)$, $\alpha(s)$ and $\gamma(s)$ are called TWISS PARAMETERS

MATRIX FORMALISM AND TWISS PARAMETERS (4/5)

- **Hill's equation with piecewise periodic constant coefficients => Assume *K* (s) to be a constant *K* over a distance *l* between the azimuthal locations s_0 and s_1: > 0, < 0 or = 0**

  - **For K > 0**

$$M\left(s_1 / s_0\right) = \begin{bmatrix} \cos\left(\sqrt{K}\, l\right) & \dfrac{1}{\sqrt{K}} \sin\left(\sqrt{K}\, l\right) \\ -\sqrt{K} \sin\left(\sqrt{K}\, l\right) & \cos\left(\sqrt{K}\, l\right) \end{bmatrix}$$

  - **For K < 0**

$$M\left(s_1 / s_0\right) = \begin{bmatrix} \cosh\left(\sqrt{|K|}\, l\right) & \dfrac{1}{\sqrt{|K|}} \sinh\left(\sqrt{|K|}\, l\right) \\ \sqrt{|K|} \sinh\left(\sqrt{|K|}\, l\right) & \cosh\left(\sqrt{|K|}\, l\right) \end{bmatrix}$$

  - **For K = 0 (drift space)**

$$M\left(s_1 / s_0\right) = \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix}$$

MATRIX FORMALISM AND TWISS PARAMETERS (5/5)

- **If the cell is made of *N* elements having transfer matrices M_1, M_2, …, M_N**

$$M_k = M\left(s_k / s_{k-1}\right)$$

=>

$$M\left(s_0\right) \equiv M\left(s_0 + L / s_0\right) = M_N \; M_{N-1} \cdots M_2 \; M_1$$

- **Noting $m_{ij}\left(s_0\right)$ the components of the matrix $M\left(s_0\right)$, the Twiss parameters at the reference point s_0 are given by**

$$\beta\left(s_0\right) = \frac{m_{12}\left(s_0\right)}{\sin\mu}$$

$$\alpha\left(s_0\right) = \frac{m_{11}\left(s_0\right) - m_{22}\left(s_0\right)}{2\sin\mu}$$

$$\gamma\left(s_0\right) = -\frac{m_{21}\left(s_0\right)}{\sin\mu}$$

$$\cos\mu = \frac{1}{2}\left[m_{11}\left(s_0\right) + m_{22}\left(s_0\right)\right]$$

THIN LENS APPROXIMATION AND FODO CELL (1/4)

- **Thin lens approximation**

$$\sqrt{K}\, l << 1 \quad \text{with} \quad K > 0$$

$$l \rightarrow 0 \quad \text{with} \quad K\, l = \text{constant}$$

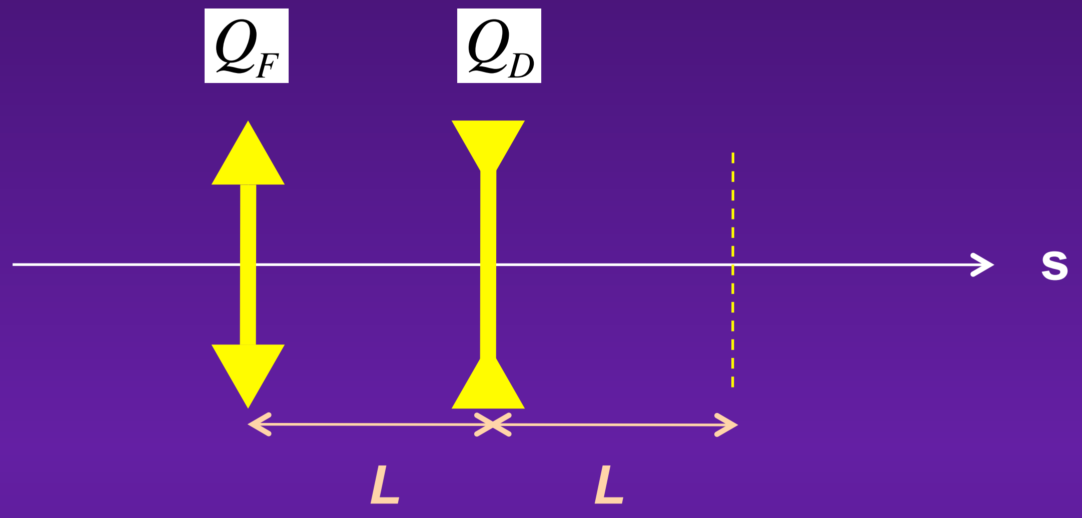
=> Transfer matrix of a thin quadrupole is

$$M\left(l/0\right) = \begin{bmatrix} 1 & 0 \\ \mp K\, l & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mp f^{-1} & 1 \end{bmatrix}$$

- => Horizontally focusing quadrupole
+ => Horizontally defocusing quadrupole

THIN LENS APPROXIMATION AND FODO CELL (2/4)

- **Transfer matrix of a symmetric thin-lens FODO cell**



$$M_{FODO}(2L/0) = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ f^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -f^{-1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{L}{f} - \frac{L^2}{f^2} & 2L + \frac{L^2}{f} \\ -\frac{L}{f^2} & 1 + \frac{L}{f} \end{pmatrix}$$

THIN LENS APPROXIMATION AND FODO CELL (3/4)

- **By comparison with the general form of a Twiss matrix**

$$M(s) = \begin{bmatrix} \cos\mu + \alpha(s)\sin\mu & \beta(s)\sin\mu \\ -\gamma(s)\sin\mu & \cos\mu - \alpha(s)\sin\mu \end{bmatrix}$$

one obtains

$$\beta_{QF}\sin\mu = 2L + \frac{L^2}{f} = 2L\left(1 + \frac{L}{2f}\right)$$

and

$$\mathrm{Tr}\left[M(s)\right] = 2\cos\mu = 2 - \frac{L^2}{f^2}$$

=>

$$\boxed{\frac{L}{2f} = \sin\left(\frac{\mu}{2}\right)}$$

=>

$$\boxed{\beta_{QF} = 2L\,\frac{1+\sin\left(\frac{\mu}{2}\right)}{\sin\mu}}$$

THIN LENS APPROXIMATION AND FODO CELL (4/4)

The betatron function at the defocusing quadrupole QD can be found by considering the DOFO cell instead of a FODO cell

$$\boxed{\beta_{QD} = 2\,L\,\frac{1-\sin\left(\frac{\mu}{2}\right)}{\sin\mu}}$$

- Stability of the FODO cell requires $\left|\cos\mu\right| \leq 1$

  <=> $\boxed{\left|\,f\,\right| \geq \frac{L}{2}}$: the stability of the FODO cell is thus obtained for distances *L* between the quadrupoles up to twice their focal length

BEAM EMITTANCE, ENVELOPE AND DIVERGENCE (1/7)

- An invariant i.e. a constant of motion, (called COURANT-SNYDER INVARIANT) can be found from the solution of the Hill's equation

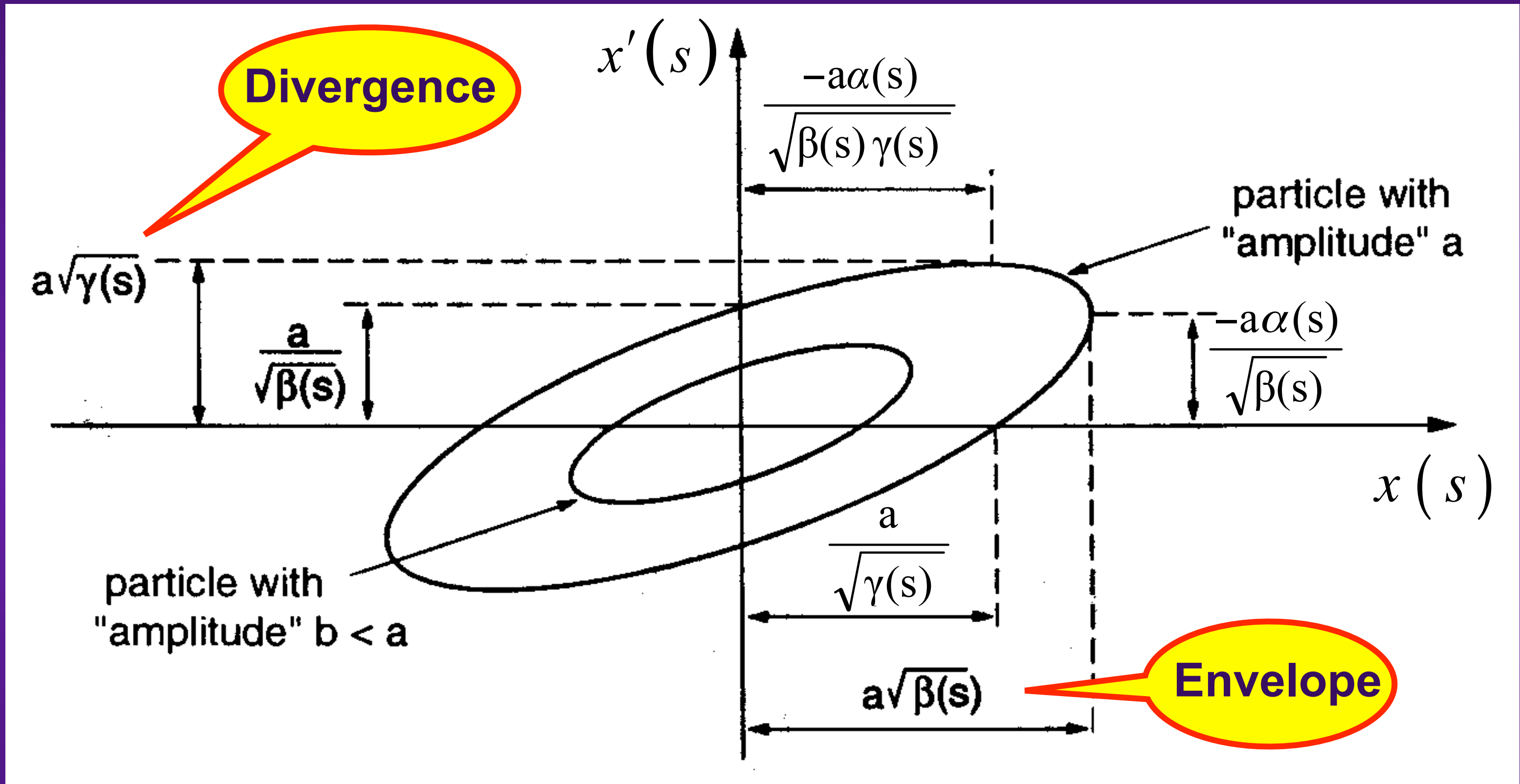
$$u(s) = \mathrm{a} \sqrt{\beta(s)} \cos\left[\mu(s) - \varphi\right]$$

=> Equation of an ellipse (motion for one particle) in the phase space plane (u, u'), with area π a^2

$$\boxed{\gamma(s)\, u(s)^2 + 2\,\alpha(s)\, u(s)\, u'(s) + \beta(s)\, u'(s)^2 = a^2}$$

"Single-particle" emittance: $\varepsilon = a^2$

BEAM EMITTANCE, ENVELOPE AND DIVERGENCE (2/7)



=> The shape and orientation of the phase plane ellipse evolve along the machine (depending on the Twiss parameters), but not its area

BEAM EMITTANCE, ENVELOPE AND DIVERGENCE (3/7)

- Transformation rule for phase ellipses through the lattice

Cosine-like solution

Sine-like solution

$$\begin{bmatrix} u(s) \\ u'(s) \end{bmatrix} = \begin{bmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{bmatrix} \begin{bmatrix} u(s_0) \\ u'(s_0) \end{bmatrix}$$

where $C(s)$ and $S(s)$ are 2 independent solutions of Hill's equation satisfying the particular initial conditions: $C(s_0) = S'(s_0) = 1$ and $C'(s_0) = S(s_0) = 0$

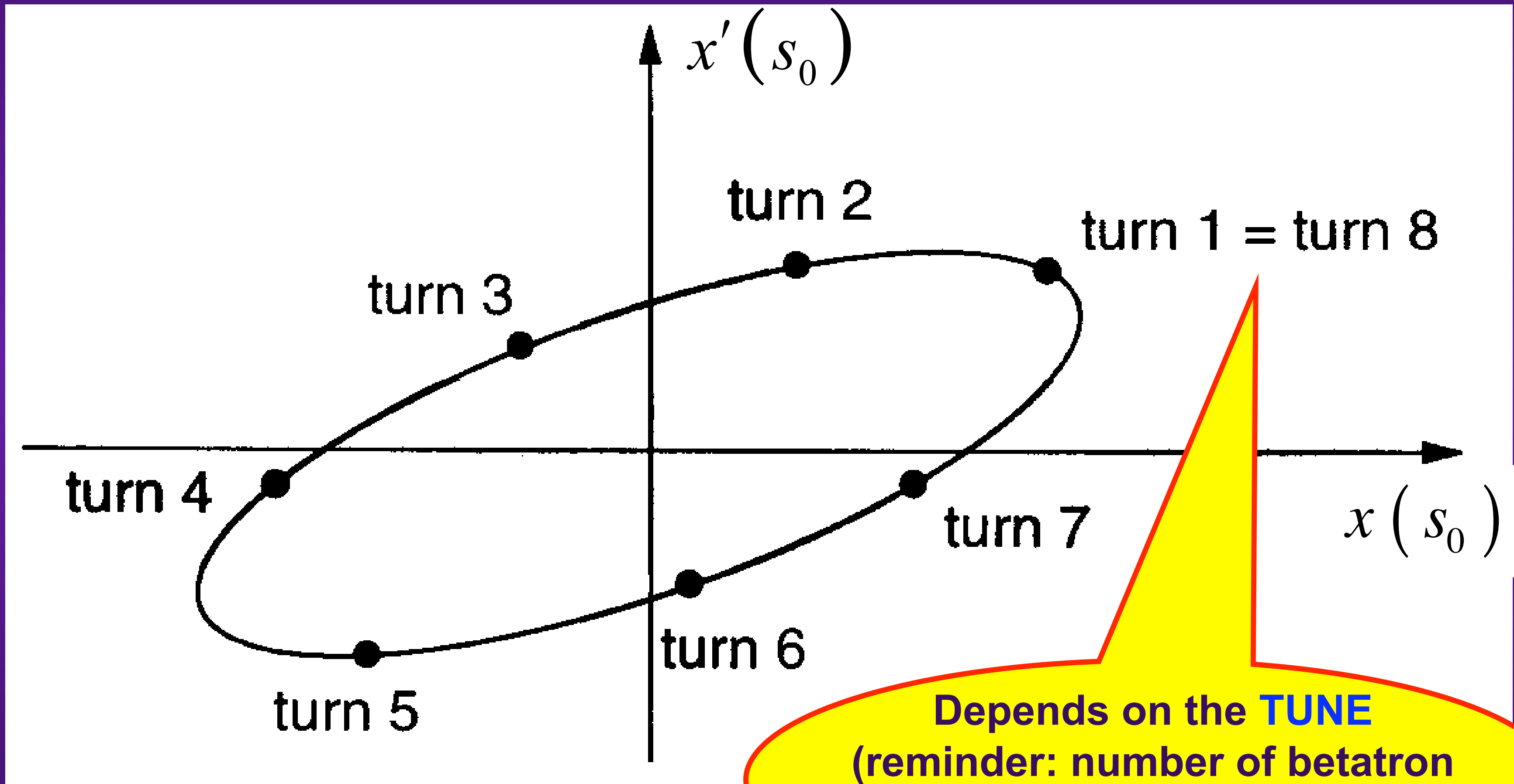
For K (s) = K > 0: $C(s) = \cos\left(\sqrt{K}\, s\right)$

$S(s) = \sin\left(\sqrt{K}\, s\right) / \sqrt{K}$

=> $$\begin{bmatrix} \beta(s) \\ \alpha(s) \\ \gamma(s) \end{bmatrix} = \begin{bmatrix} C(s)^2 & -2S(s)C(s) & S(s)^2 \\ -C(s)C'(s) & S'(s)C(s) + S(s)C'(s) & -S(s)S'(s) \\ C'(s)^2 & -2S'(s)C'(s) & S'(s)^2 \end{bmatrix} \begin{bmatrix} \beta(s_0) \\ \alpha(s_0) \\ \gamma(s_0) \end{bmatrix}$$

BEAM EMITTANCE, ENVELOPE AND DIVERGENCE (4/7)

Stroboscopic representation or POINCARÉ MAPPING



Depends on the TUNE (reminder: number of betatron oscillations per machine revolution)

BEAM EMITTANCE, ENVELOPE AND DIVERGENCE (5/7)

- **BEAM EMITTANCE = Measure of the spread in phase space of the points representing beam particles ⇒ 3 definitions**

  1) **In terms of the phase plane "amplitude" a_q enclosing q % of the particles**

$$\iint_{\substack{\text{ellipse of} \\ \text{"amplitude" } a_q}} dx\, dx' = \pi\, \varepsilon_x^{(q\%)}$$

[mm mrad] or [μm]

  2) **In terms of the 2nd moments of the particle distribution**

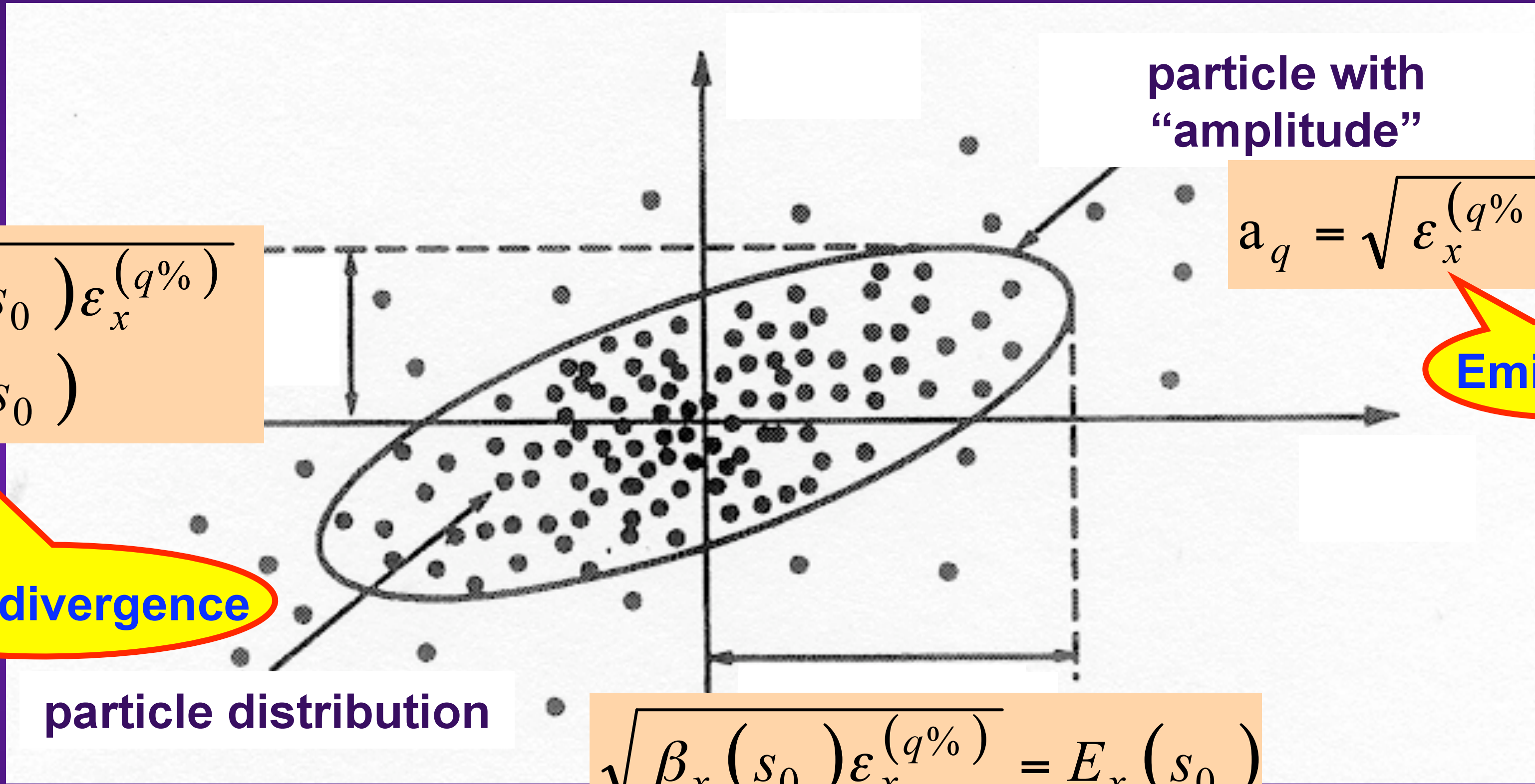
$$\varepsilon_x^{(\text{stat})} \equiv \sqrt{< x^2 >< x'^2 > - < x\, x' >^2}$$

Determinant of the covariance matrix

  3) **In terms of σ_x the standard deviation of the particle distribution in real space (= projection onto the x-axis)**

$$\boxed{\varepsilon_x^{(\sigma_x)} \equiv \frac{\sigma_x^2}{\beta_x}}$$

BEAM EMITTANCE, ENVELOPE AND DIVERGENCE (6/7)



$a_q = \sqrt{\varepsilon_x^{(q\%)}}$

Emittance

$\sqrt{\gamma_x(s_0)\,\varepsilon_x^{(q\%)}} = A_x(s_0)$

Beam divergence

$\sqrt{\beta_x(s_0)\,\varepsilon_x^{(q\%)}} = E_x(s_0)$

Beam envelope

The *β*-function reflects the size of the beam and depends only on the lattice

BEAM EMITTANCE, ENVELOPE AND DIVERGENCE (7/7)

- MACHINE mechanical (i.e. from the vacuum chamber) ACCEPTANCE or APERTURE = Maximum beam emittance

- NORMALIZED BEAM EMITTANCE

Relativistic factors

$$\boxed{\varepsilon_{x,norm}^{(\sigma_x)} = \beta_r \, \gamma_r \, \varepsilon_x^{(\sigma_x)}}$$

⇒ The normalized emittance is conserved during acceleration (in the absence of collective effects…)

- ADIABATIC DAMPING: As $\beta_r \gamma_r$ increases proportionally to the particle momentum p, the (physical) emittance decreases as $1 / p$

- However, many phenomena may affect (increase) the emittance

- An important challenge in accelerator technology is to preserve beam emittance and even to reduce it (by COOLING)

DISPERSION (1/10)

- In practice, particle beams have a finite dispersion of momenta about the ideal momentum p_0. A particle with momentum $p \neq p_0$ will perform betatron oscillations around **A DIFFERENT CLOSED ORBIT** from that of the reference particle

$$x'' + \left(K_0 + \frac{1}{\rho_x^2} \right) x = \frac{\delta}{\rho_x}$$

$$y'' - \left(K_0 - \frac{1}{\rho_y^2} \right) y = \frac{\delta}{\rho_y}$$

=> $$u'' + K(s)\, u = \frac{\delta}{\rho(s)}$$

=> $$\boxed{u(s) = u_\delta(s) + u_\beta(s)}$$

Displacement of the closed orbit for the off-momentum particle from the on-momentum particle

Betatron oscillation around the off-momentum orbit

DISPERSION (2/10)

$$\boxed{u_\delta(s) = D(s)\,\delta}$$

$D(s)$ **is called the DISPERSION FUNCTION**

- **The dispersion function satisfies the equation**

$$D'' + K(s)\,D = \frac{1}{\rho(s)}$$

and can be found to be given by

$$\boxed{D(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi Q)} \int_s^{s+C} dt\, \frac{\sqrt{\beta(t)}}{\rho(t)} \cos\left[\mu(t) - \mu(s) - \pi Q\right]}$$

DISPERSION (3/10)

=> i) The dispersion function depends on all the bending magnets in the accelerator

ii) To get stable off-momentum orbits, the machine tune *Q* must not be an integer (otherwise resonance effect will occur)

- Applying the same method as the one used for betatron oscillations, the dispersion function can be derived by applying the matrix formalism (instead of using the previous formula) => 3 × 3 transfer matrices

$$\boxed{\begin{bmatrix} u(s) \\ u'(s) \\ \delta \end{bmatrix} = M(s/s_0) \begin{bmatrix} u(s_0) \\ u'(s_0) \\ \delta \end{bmatrix}}$$

$$M(s_0) \equiv M(s_0 + L / s_0) = M_N \; M_{N-1} \cdots M_2 \; M_1$$

$$= \begin{bmatrix} m_{11}(s_0) & m_{12}(s_0) & m_{13}(s_0) \\ m_{21}(s_0) & m_{22}(s_0) & m_{23}(s_0) \\ 0 & 0 & 1 \end{bmatrix}$$

Usual 2 × 2 transfer matrix for betatron oscillations

DISPERSION (4/10)

It can be shown that

$$\begin{bmatrix} D(s) \\ D'(s) \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11}(s_0) & m_{12}(s_0) & m_{13}(s_0) \\ m_{21}(s_0) & m_{22}(s_0) & m_{23}(s_0) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} D(s) \\ D'(s) \\ 1 \end{bmatrix}$$

=> The dispersion function and its derivative can be found at any location s_0 around the ring

$$D(s_0) = \frac{m_{13}(s_0)\left[1 - m_{22}(s_0)\right] + m_{12}(s_0)\, m_{23}(s_0)}{2 - m_{11}(s_0) - m_{22}(s_0)}$$

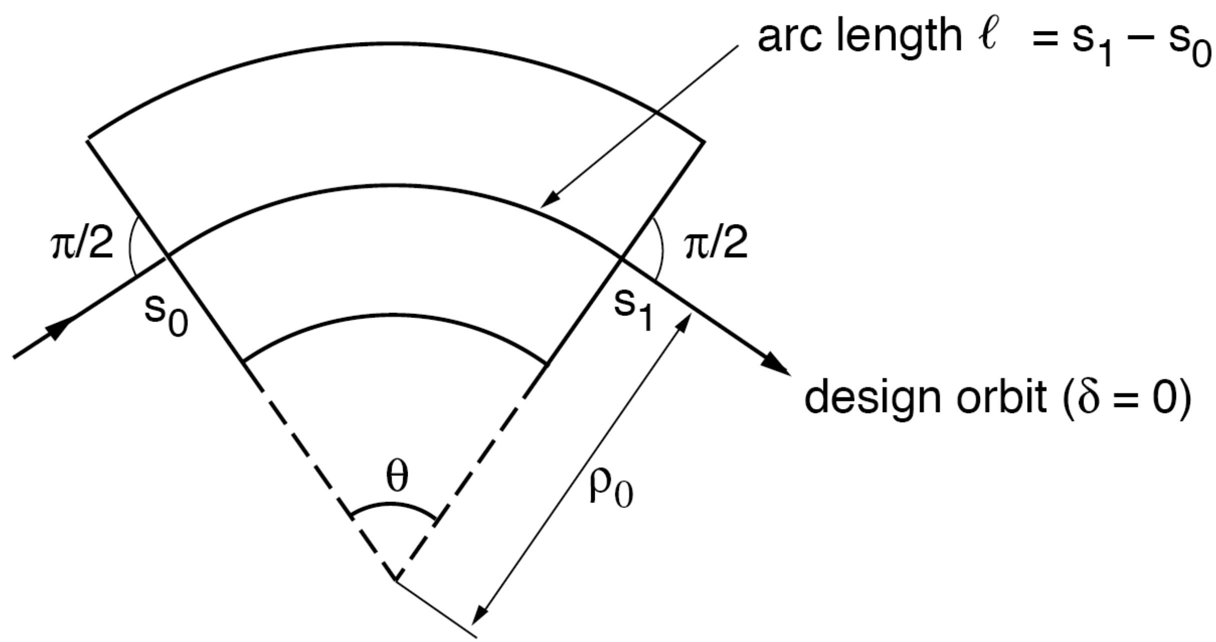
$$D'(s_0) = \frac{m_{23}(s_0)\left[1 - m_{11}(s_0)\right] + m_{21}(s_0)\, m_{13}(s_0)}{2 - m_{11}(s_0) - m_{22}(s_0)}$$

DISPERSION (5/10)

- **Example of a dipole sector magnet**

$$K(s) = \frac{1}{\rho_0^2}$$

$$\vartheta = \frac{l}{\rho_0}$$



=> $C(s) = \cos\left(\frac{s - s_0}{\rho_0}\right)$ $S(s) = \rho_0 \sin\left(\frac{s - s_0}{\rho_0}\right)$

DISPERSION (6/10)

=>

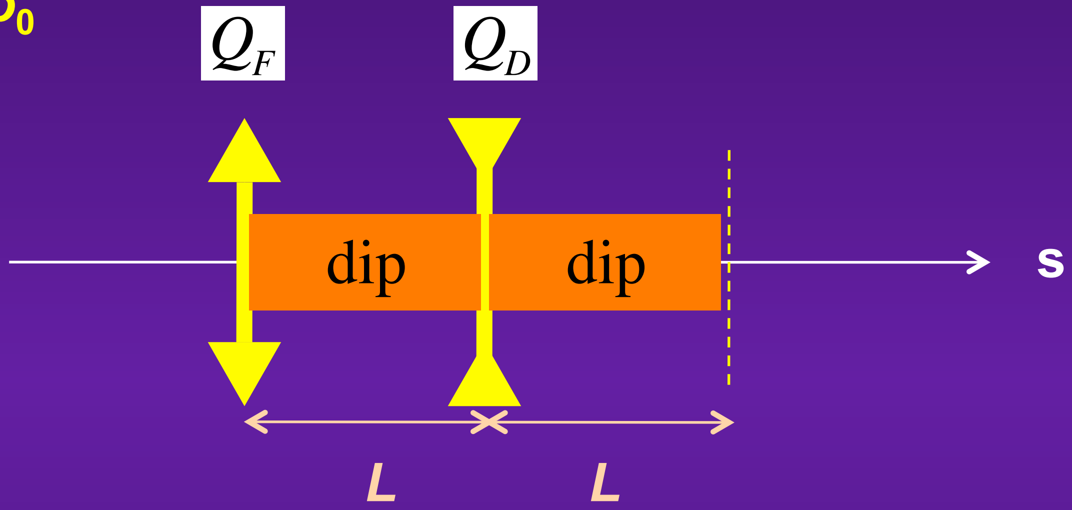
$$M\left(s_1 / s_0\right) = \begin{bmatrix} \cos\vartheta & \rho_0 \sin\vartheta & \rho_0\left(1-\cos\vartheta\right) \\ -\dfrac{\sin\vartheta}{\rho_0} & \cos\vartheta & \sin\vartheta \\ 0 & 0 & 1 \end{bmatrix}$$

- Note that the 3 × 3 transfer matrix for synchrotron magnets (= combined-function dipole-quadrupole magnets) can be derived similarly, replacing $1/\rho_0$ by $\sqrt{K}$ where

$$K = K_0 + \frac{1}{\rho_0^2}$$

DISPERSION (7/10)

- **Example of the thin-lens FODO lattice with cell length 2 *L*, in which the drift spaces are replaced by dipole sector magnets of length *L* and bending radius ρ_0**



$$M_{FdipDdip}\left(2\,L\,/\,0\right) = M_{dip}\; M_{QD}\; M_{dip}\; M_{QF}$$

$$M_{QF,D} = \begin{bmatrix} 1 & 0 & 0 \\ \mp f^{-1} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_{dip} = \begin{bmatrix} 1 & L & \dfrac{L^2}{2\rho_0} \\ 0 & 1 & \dfrac{L}{\rho_0} \\ 0 & 0 & 1 \end{bmatrix}$$

DISPERSION (8/10)

=> $$M_{FdipDdip}(2L/0)=\begin{bmatrix} 1-\frac{L}{f}-\frac{L^2}{f^2} & 2L+\frac{L^2}{f} & \frac{L^2}{2\rho_0}\left(4+\frac{L}{f}\right) \\ -\frac{L}{f^2} & 1+\frac{L}{f} & \frac{L}{2\rho_0}\left(4+\frac{L}{f}\right) \\ 0 & 0 & 1 \end{bmatrix}$$

- **Using the definition of the dispersion function, it can be computed at the position s_0 (of the focusing quad QF)**

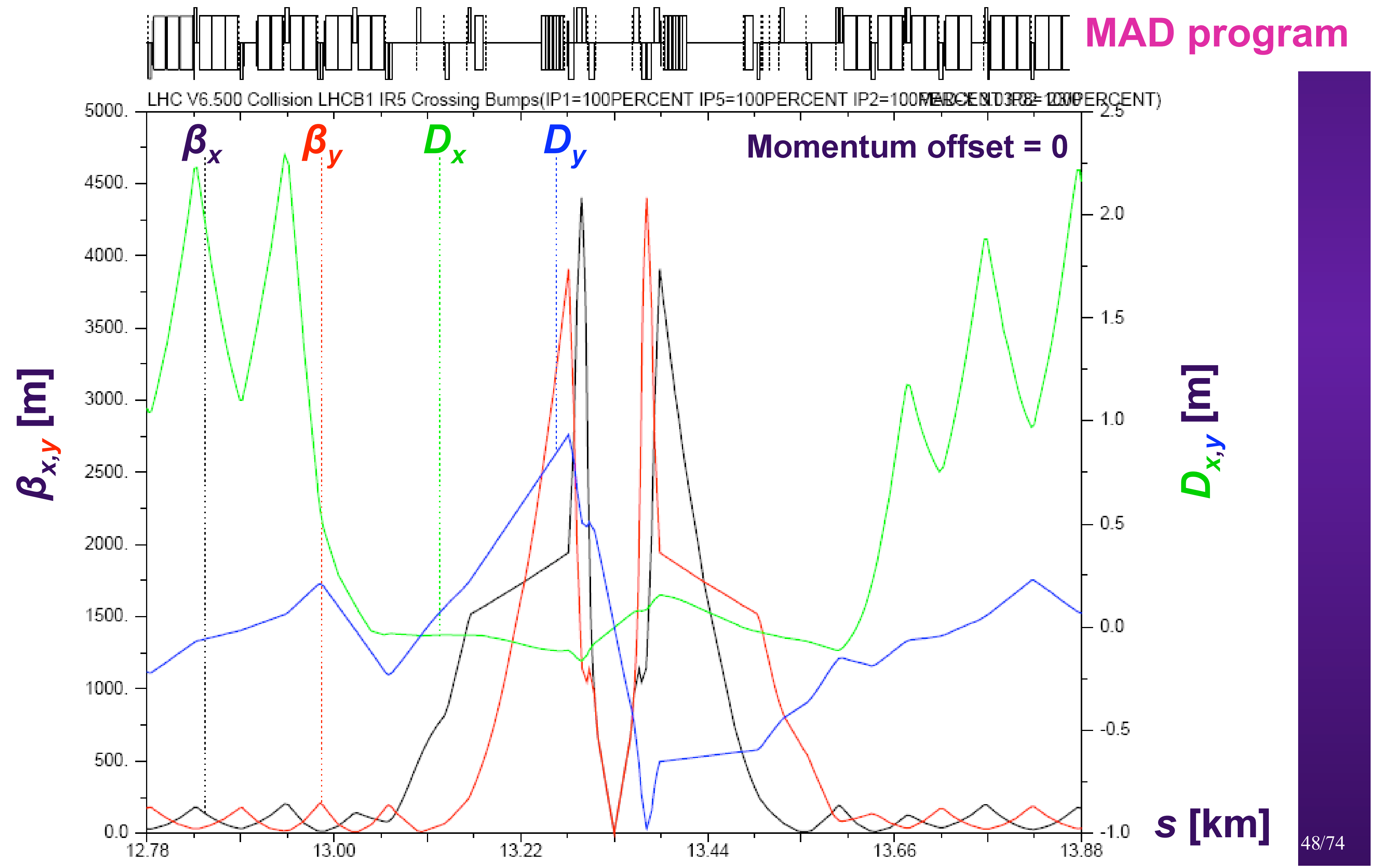
$$D_{QF}=\frac{4f^2}{\rho_0}\left(1+\frac{L}{4f}\right)$$

- **Similarly, the dispersion in the defocusing quad QD is derived by inverting the QF and QD in the cell**

$$D_{QD}=\frac{4f^2}{\rho_0}\left(1-\frac{L}{4f}\right)$$

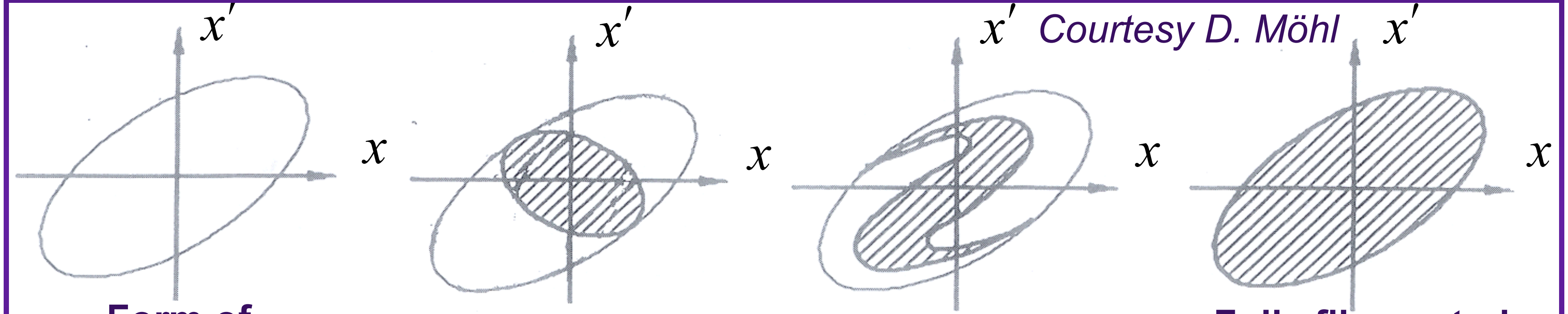
DISPERSION (9/10)

- **Example of LHC optics for the Interaction Point 5 (CMS) in collision**



DISPERSION (10/10)

- **BETATRON MATCHING = The phase space ellipses at the injection (ejection) point of the circular machine, and the exit (entrance) of the beam transport line, should be homothetic. To do this, the Twiss parameters are modified using quadrupoles. If the ellipses are not homothetic, there will be a dilution (i.e. a BLOW-UP) of the emittance**



Courtesy D. Möhl

Form of matched ellipse | **Unmatched beam** | **Filamenting beam** | **Fully filamented beam**

- **DISPERSION MATCHING = D_x and D'_x should be the same at the injection (ejection) point of the circular machine, and the exit (entrance) of the beam transport line. If there are different, there will be also a BLOW-UP, but due to a missteering (because the beam is not injected on the right orbit)**

NORMALIZED FLOQUET'S COORDINATES (1/3)

- **Normalized (Floquet's) coordinates**

$$\eta = \frac{x}{\sqrt{\beta_x(s)}} \qquad \zeta = \frac{y}{\sqrt{\beta_y(s)}} \qquad \phi = \frac{1}{Q_{x,y}} \int_{s_0}^{s} \frac{dt}{\beta_{x,y}(t)}$$

=> Perturbed (e.g. horizontal) Hill's equation

General perturbation term

$$x'' + K(s)\,x = p(x,y,s)$$

becomes

$$\ddot{\eta} + Q^2\,\eta = Q^2\,\beta^{3/2}\,p(\eta,\phi) \qquad \dot{\eta} = \frac{d\eta}{d\phi}$$

NORMALIZED FLOQUET'S COORDINATES (2/3)

- When the perturbation vanishes => $\ddot{\eta} + Q^2 \, \eta = 0$

Harmonic oscillator with frequency *Q*

=> $\eta(\phi) = a \cos\left[\mu(\phi) - \varphi\right]$

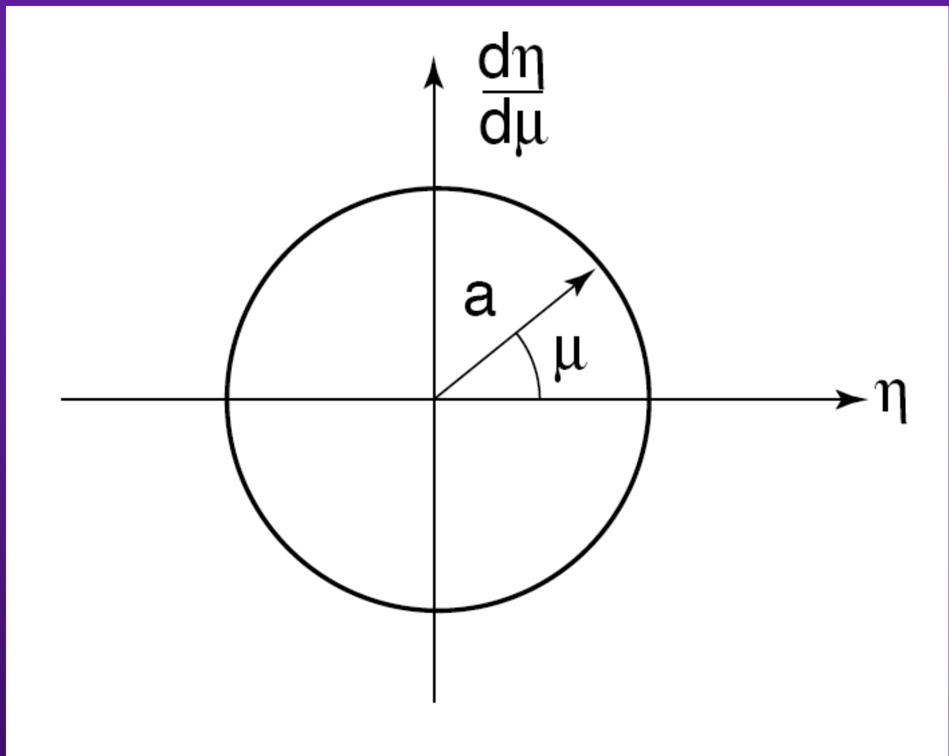
- **Phase advance** $\mu(\phi) = Q\,\phi = \int_{s_0}^{s} \frac{dt}{\beta(t)}$

- **Particle trajectory in phase space** $\boxed{\eta^2 + \left(\frac{d\eta}{d\mu}\right)^2 = a^2}$

NORMALIZED FLOQUET'S COORDINATES (3/3)

=> The particle trajectory in the phase plane (*η* , d*η* / d*μ*) is thus a circle of radius equal to the amplitude *a* of the oscillation

=> The phase advances by 2 π every betatron oscillation or by 2 π Q every machine revolution



CHROMATICITY AND ITS CORRECTION (1/17)

- **Chromaticity**
  - **Equations of motion of an off-momentum particle (without skew magnets and sextupoles and in a zero-curvature region)**

$$x'' + K_0 \left(1 - \delta\right) x = 0$$

$$y'' - K_0 \left(1 - \delta\right) y = 0$$

  - **Chromaticity = Variation of the tune with the momentum**

$$\boxed{Q' = \frac{\Delta Q}{\delta}}$$

  - **Relative chromaticity** $\boxed{\xi = \frac{Q'}{Q}}$

CHROMATICITY AND ITS CORRECTION (2/17)

$$K_0 = \frac{e}{p_0} \frac{\partial B_y}{\partial x}$$

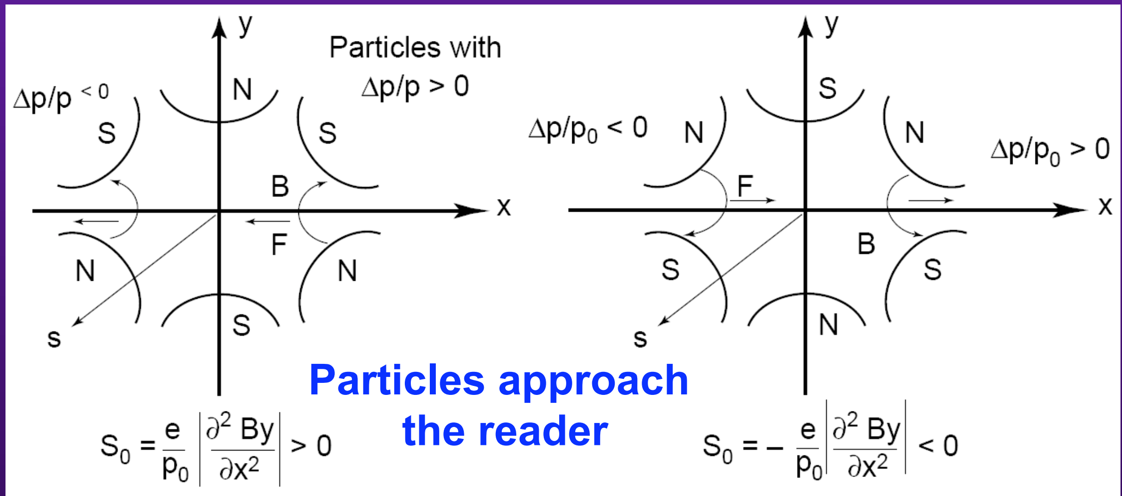
$$K(\delta) = \frac{e}{p} \frac{\partial B_y}{\partial x}$$

=>

$$\boxed{\frac{1}{f(\delta)} = \frac{K_0 \, l}{1+\delta}}$$

- **Once the particles are spread by momentum in a region with dispersion, we can apply focusing corrections depending on the momentum using a sextupole magnet**

SEXTUPOLE =
1st nonlinear magnet



CHROMATICITY AND ITS CORRECTION (3/17)

=> The sextupole is focusing for the higher-momentum particles and defocusing for the lower-momentum particles => It can be used to correct the chromatic focusing errors in a region with non-zero dispersion

- Let's consider the equations of motion including the contributions from the regular sextupole magnets

$$x'' + K_0 \left(1 - \delta \right) x = - \frac{1}{2} S_0 \left(x^2 - y^2 \right)$$

$$y'' - K_0 \left(1 - \delta \right) y = S_0 \, x \, y$$

- Using $x(s) = x_\delta(s) + x_\beta(s)$, $y(s) = y_\beta(s)$ yields

CHROMATICITY AND ITS CORRECTION (4/17)

$$x_\beta'' + K_0 \, x_\beta = \left(K_0 - S_0 \, D_x \right) x_\beta \, \delta$$

$$y_\beta'' - K_0 \, y_\beta = -\left(K_0 - S_0 \, D_x \right) y_\beta \, \delta$$

(discarding the terms which do not depend on the betatron motion, because they do not contribute to the chromatic tune shift, and ignoring the non-chromatic terms of 2nd order)

- **Introducing the normalized coordinates, yields**

CHROMATICITY AND ITS CORRECTION (5/17)

$$\ddot{\eta} + Q_x^2 \, \eta = Q_x \; p_{x\delta}\left(\phi\right) \eta$$

$$\ddot{\zeta} + Q_y^2 \, \zeta = - Q_y \; p_{y\delta}\left(\phi\right) \zeta$$

$$p_{x\delta}\left(\phi\right) = Q_x \; \beta_x^2 \left[K_0\left(\phi\right) - S_0\left(\phi\right) D_x \right] \delta$$

$$p_{y\delta}\left(\phi\right) = Q_y \; \beta_y^2 \left[K_0\left(\phi\right) - S_0\left(\phi\right) D_x \right] \delta$$

Periodic functions with period 2 π

CHROMATICITY AND ITS CORRECTION (6/17)

- **Expanding the perturbations in Fourier series yields**

$$p_{x\delta}(\phi) = \sum_{m=-\infty}^{+\infty} \hat{p}_{x\delta}(m)\, e^{jm\phi}$$

$$= \hat{p}_{x\delta}(0) + \sum_{m \neq 0} \hat{p}_{x\delta}(m)\, e^{jm\phi}$$

with

$$\hat{p}_{x\delta}(m) = \frac{1}{2\pi} \int_0^{2\pi} d\phi\, p_{x\delta}(\phi)\, e^{-jm\phi}$$

Fourier coefficients

CHROMATICITY AND ITS CORRECTION (7/17)

=> $$\ddot{\eta} + \tilde{Q}_x^2(\phi)\,\eta = 0$$ with

$$\tilde{Q}_x^2(\phi) = Q_x^2 \left[1 - \frac{\hat{p}_{x\delta}(0)}{Q_x} \right] - Q_x \sum_{m \neq 0} \hat{p}_{x\delta}(m)\, e^{\,j m \phi}$$

Averages to 0 over 1 period 2 π

- Averaging over a period gives the static tune shift due to chromatic effect

$$\ddot{\eta} + \left(Q_x + \Delta Q_x \right)^2 \eta = 0$$ with $$\boxed{\Delta Q_x = -\frac{\hat{p}_{x\delta}(0)}{2}}$$

CHROMATICITY AND ITS CORRECTION (8/17)

=>

$$\Delta Q_x = -\frac{1}{4\pi}\int_0^{2\pi} d\phi\, p_{x\delta}(\phi)$$

$$= -\frac{\delta}{4\pi}\int_{s_0}^{s_0+C} ds\, \beta_x(s)\left[K_0(s) - S_0(s)\, D_x(s)\right]$$

$$\Delta Q_y = \frac{\delta}{4\pi}\int_{s_0}^{s_0+C} ds\, \beta_y(s)\left[K_0(s) - S_0(s)\, D_x(s)\right]$$

=>

$$\xi_x = -\frac{1}{4\pi Q_x}\int_{s}^{s+C} dt\, \beta_x(t)\left[K_0(t) - S_0(t)\, D_x(t)\right]$$

$$\xi_y = +\frac{1}{4\pi Q_y}\int_{s}^{s+C} dt\, \beta_y(t)\left[K_0(t) - S_0(t)\, D_x(t)\right]$$

CHROMATICITY AND ITS CORRECTION (9/17)

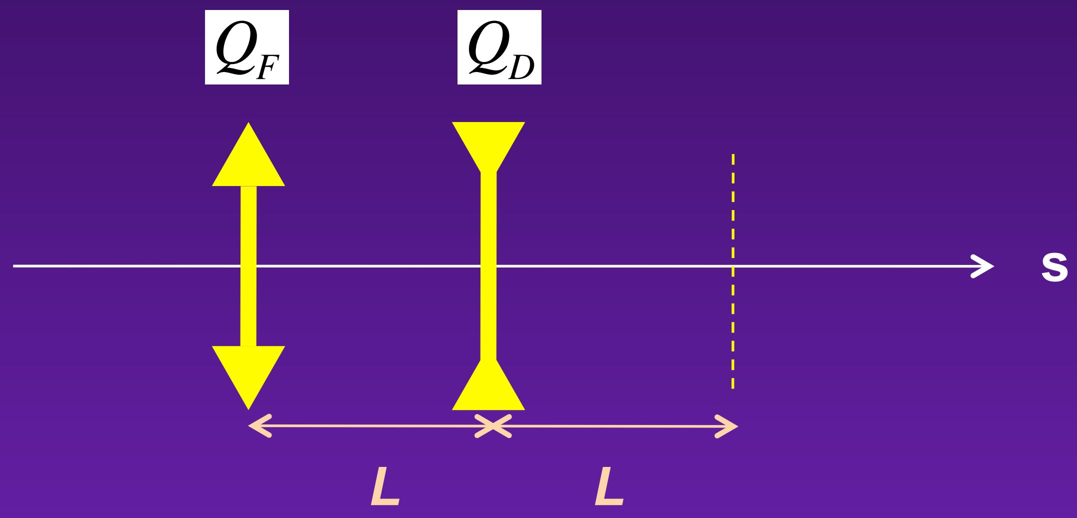
- **The contribution to chromaticity arising from pure quadrupole elements (and also pure dipoles) is called natural chromaticity => The natural chromaticities of a lattice are**

$$\xi_x^{natural} = - \frac{1}{4 \pi Q_x} \int_s^{s+C} d\,t \, \beta_x(t) \, K_0(t)$$

$$\xi_y^{natural} = + \frac{1}{4 \pi Q_y} \int_s^{s+C} d\,t \, \beta_y(t) \, K_0(t)$$

CHROMATICITY AND ITS CORRECTION (10/17)

- **Natural chromaticity of a FODO cell**



$$\xi_x^{natural} = -\frac{1}{4\pi Q_x} \int_s^{s+C} dt\, \beta_x(t)\, K_0(t)$$

$$= -\frac{1}{4\pi Q_x} \left(\beta_x^{QF} \int ds\, K^{QF} + \beta_x^{QD} \int ds\, K^{QD} \right)$$

$$= -\frac{1}{4\pi Q_x} \left(\beta_x^{QF}\, K^{QF}\, l + \beta_x^{QD}\, K^{QD}\, l \right)$$

CHROMATICITY AND ITS CORRECTION (11/17)

=>

$$\xi_x^{natural} = -\frac{1}{4\,\pi\,Q_x\,f}\left(\beta_x^{QF} - \beta_x^{QD}\right)$$

with

$$\frac{1}{f} = K^{QF}\,l = -\,K^{QD}\,l$$

- We saw previously that

$$\beta_x^{QF} = 2\,L\,\frac{1+\sin\left(\frac{\mu}{2}\right)}{\sin\mu} \qquad \beta_x^{QD} = 2\,L\,\frac{1-\sin\left(\frac{\mu}{2}\right)}{\sin\mu} \qquad \frac{1}{f} = \frac{2}{L}\sin\left(\frac{\mu}{2}\right)$$

=>

$$\xi_x^{natural} = -\frac{1}{\pi\,Q_x}\tan\left(\frac{\mu}{2}\right)$$

CHROMATICITY AND ITS CORRECTION (12/17)

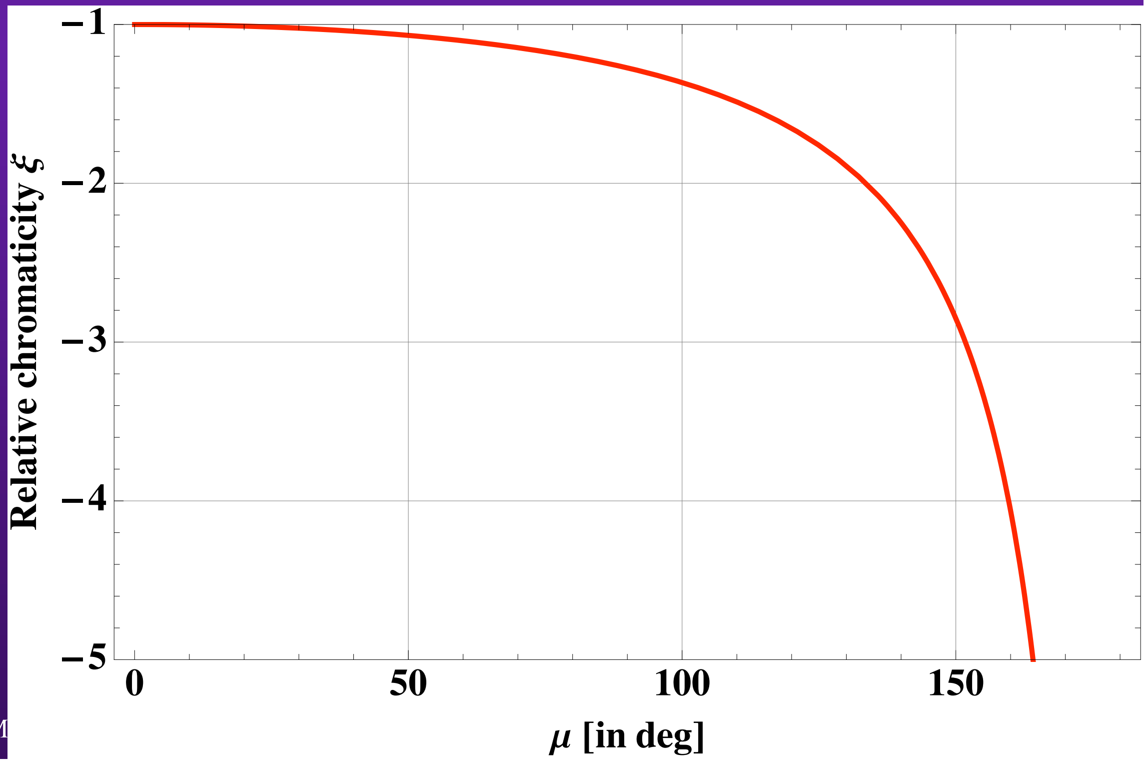
- **If the full lattice is composed of N_{cell} similar FODO cells**

$$\xi_x^{natural} = -\frac{N_{cell}}{\pi\, Q_x} \tan\left(\frac{\mu}{2}\right)$$

Furthermore, $$Q_x = \frac{N_{cell}\, \mu}{2\, \pi}$$ **=>** $$\xi_x^{natural} = -\frac{2}{\mu} \tan\left(\frac{\mu}{2}\right)$$

Similarly

$$\xi_y^{natural} = -\frac{2}{\mu} \tan\left(\frac{\mu}{2}\right)$$



CHROMATICITY AND ITS CORRECTION (13/17)

- **The natural chromaticities of a synchrotron made up of FODO cells are negative. For instance, if the phase advance per cell is π / 2, then**

$$\xi_x^{natural} = \xi_y^{natural} = -\frac{4}{\pi} \approx -1.3$$

CHROMATICITY AND ITS CORRECTION (14/17)

- More generally, the natural chromaticities are always negative since for the higher-momentum particles the focusing is less effective and then the tune is reduced

- The control of the chromaticity (using SEXTUPOLE magnets) is very important for 2 reasons
  - Avoid shifting the beam on resonances due to changes induced by chromatic effects
  - Prevent some transverse coherent (head-tail) instabilities

=> CHROMATICITY CORRECTION / CONTROL USUALLY REQUIRED

CHROMATICITY AND ITS CORRECTION (15/17)

- **Chromaticity correction**
  - The chromaticity equations suggest the insertion of sextupoles close to each quadrupole, where the dispersion function is non-zero, in order to correct the chromaticity

$$\xi_x = -\frac{1}{4\pi Q_x} \int_s^{s+C} dt \, \beta_x(t) \left[K_0(t) - S_0(t) \, D_x(t) \right]$$

=> $$\underbrace{S_0 \, l_s}_{\text{Length of sextupole}} = \frac{K_0 \, l_Q}{D_x}$$

Dispersion at the sextupole location

Length of quadrupole

Length of sextupole

  - However, such localized corrections are seldom feasible

CHROMATICITY AND ITS CORRECTION (16/17)

- A standard way of adjusting both the horizontal and vertical chromaticities is to use families of sextupoles with moderate strength, distributed around the ring

$$\xi_x = \xi_x^{natural} + \frac{1}{4 \pi Q_x} \int_s^{s+C} dt \, \beta_x(t) \, S_0(t) \, D_x(t)$$

$$\xi_y = \xi_y^{natural} - \frac{1}{4 \pi Q_y} \int_s^{s+C} dt \, \beta_y(t) \, S_0(t) \, D_x(t)$$

- Using the thin-lens approximation, one can write

$$\xi_x = \xi_x^{natural} + \frac{1}{4 \pi Q_x} \sum_{i=1}^{N} dt \, \beta_{xi} \, S_{0i} \, D_{xi} \, l_{si}$$

$$\xi_y = \xi_y^{natural} - \frac{1}{4 \pi Q_y} \sum_{i=1}^{N} dt \, \beta_{yi} \, S_{0i} \, D_{xi} \, l_{si}$$

CHROMATICITY AND ITS CORRECTION (17/17)

=> The sextupole strengths are obtained by solving a linear system of equations. Assuming only 2 sextupoles in the ring, yields

$$S_{01} = -\frac{4\pi}{D_{x1}\, l_s}\left(\frac{\beta_{y2}\, Q_x\, \Delta\xi_x + \beta_{x2}\, Q_y\, \Delta\xi_y}{\beta_{x1}\, \beta_{y2} - \beta_{x2}\, \beta_{y1}}\right)$$

$$S_{02} = \frac{4\pi}{D_{x2}\, l_s}\left(\frac{\beta_{y1}\, Q_x\, \Delta\xi_x + \beta_{x1}\, Q_y\, \Delta\xi_y}{\beta_{x1}\, \beta_{y2} - \beta_{x2}\, \beta_{y1}}\right)$$

Assuming the same length for both sextupoles l_s

NONLINEARITIES AND RESONANCES (1/5)

- In the presence of extra (NONLINEAR) FORCES, the Hill's equation takes the general form

$$x''(s) + K_x(s)\,x(s) = P_x(x, y, s)$$

Any perturbation

- Perturbation terms in the equation of motion may lead to UNSTABLE motion, called RESONANCES, when the perturbating field acts in synchronism with the particle oscillations

- A multipole of *n*th order is said to generate resonances of order *n*. Resonances below the 3rd order (i.e. due to dipole and quadrupole field errors for instance) are called LINEAR RESONANCES. The NONLINEAR RESONANCES are those of 3rd order and above

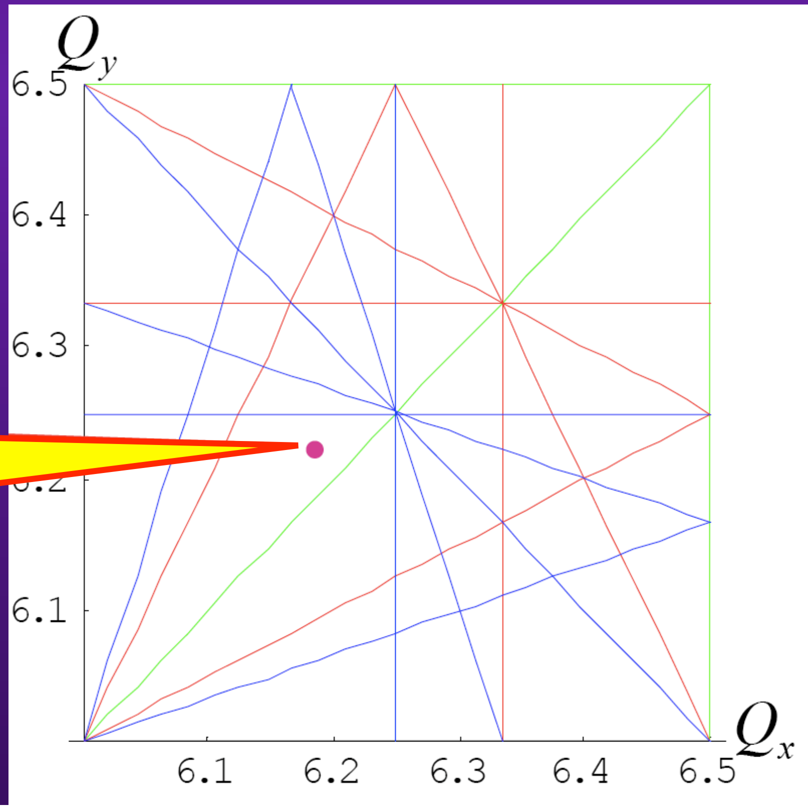
NONLINEARITIES AND RESONANCES (2/5)

- **General RESONANCE CONDITIONS** $M\,Q_x + N\,Q_y = P$

  where *M*, *N* and *P* are integers, *P* being non-negative, |M| + |N| is the order of the resonance and *P* is the order of the perturbation harmonic

- **Plotting the resonance lines for different values of *M*, *N*, and *P* in the (Q_x, Q_y) plane yields the so-called RESONANCE or TUNE DIAGRAM**

This dot in the tune diagram is called the WORKING POINT (case of the PS, here)



NONLINEARITIES AND RESONANCES (3/5)

- **RESONANCE WIDTH = Band with some thickness around every resonance line in the resonance diagram, in which the motion may be unstable, depending on the oscillation amplitude**

- **STOPBAND = Resonance width when the resonance is linear (i.e. below the 3rd order), because the entire beam becomes unstable if the operating point (Q_x, Q_y) reaches this region of tune values**

- **DYNAMIC APERTURE = Largest oscillation amplitude which is stable in the presence of nonlinearities**

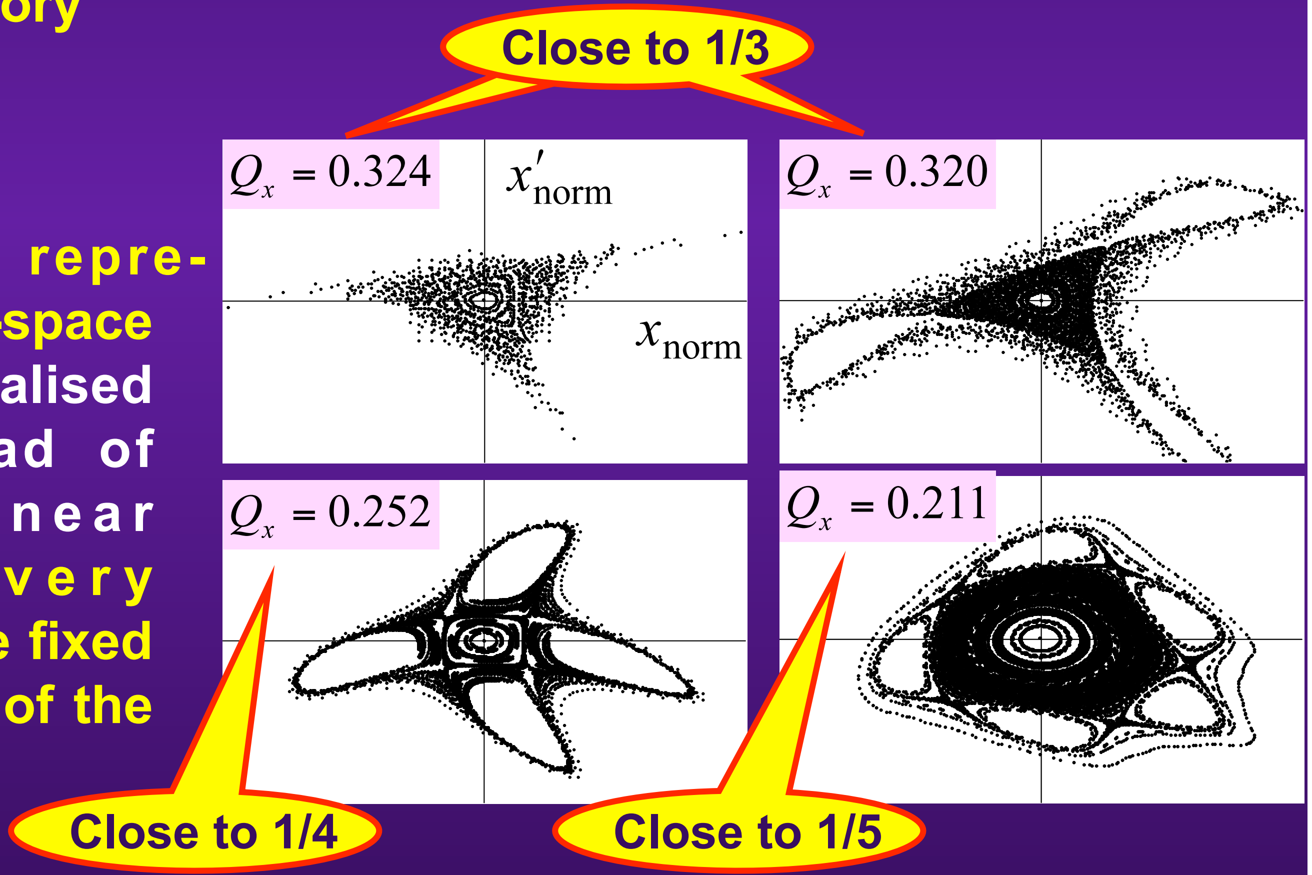
- **TRACKING: In general, the equations of motion in the presence of nonlinear fields are untractable for any but the simplest situations. Tracking consists to simulate (using computer programs such as MAD) particle motion in circular accelerators in the presence of nonlinear fields**

NONLINEARITIES AND RESONANCES (4/5)

- **KICK MODEL: Any nonlinear magnet is treated in the "point-like" approximation (i.e. the particle position is assumed not to vary as the particle traverses the field), the motion in all other elements of the lattice is assumed to be linear. Thus, at each turn the local magnetic field gives a "kick" to the particle, deflecting it from its unperturbed trajectory**

- **HENON MAPPING**

  = Stroboscopic representation of phase-space trajectories (normalised ⇒ circles instead of ellipses for linear motion) on every machine turn at the fixed azimuthal position of the perturbation



NONLINEARITIES AND RESONANCES (5/5)

- **SEPARATRICES define boundaries between stable motion (bounded oscillations) and unstable motion (expanding oscillations)**

- **The 3rd order resonance is a drastic (unstable) one because the particles which go onto this resonance are lost**

- **(STABLE) ISLANDS: For the higher order resonances (e.g. 4th and 5th) stable motions are also possible in (stable) islands. There are 4 stable islands when the tune is closed to a 4th order resonance and 5 when it is closed to a 5th order resonance**