LONGITUDINAL BEAM DYNAMICS

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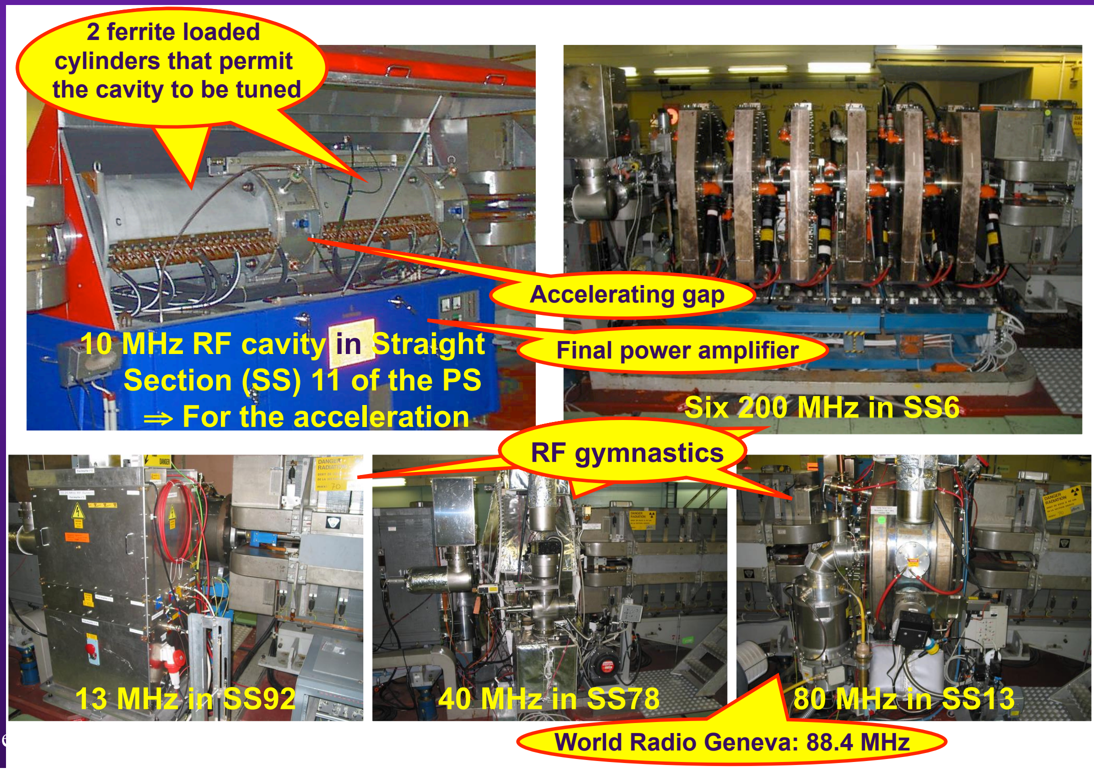
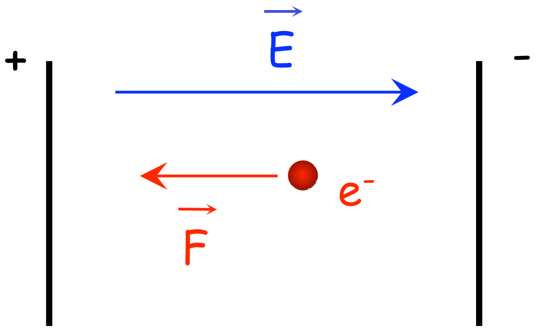
INTRODUCTION

- See course I gave at the JUAS (Joint Universities Accelerator School) in 2011-12-13 on my web page (http://emetral.web.cern.ch/emetral/, Section VI)) => Also exercises, exams and corrections of the exams
- Selection of slides of this course are presented here
- RF (Radio-Frequency) cavities are used to accelerate / decelerate and manipulate ("RF gymnastics") the particles



Constant electric field



$$\frac{\mathrm{d}\vec{p}}{\mathrm{dt}} = -e\,\vec{E}$$

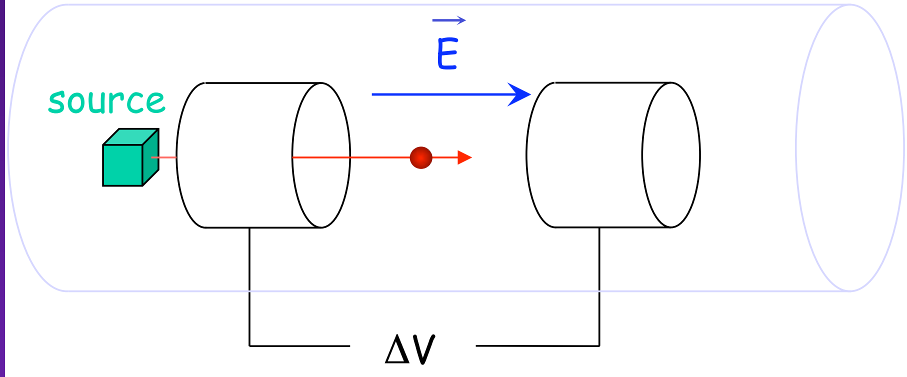
1) Direction of the force always parallel to the field

2) Trajectory can be modified, velocity also ⇒ momentum and energy can be modified

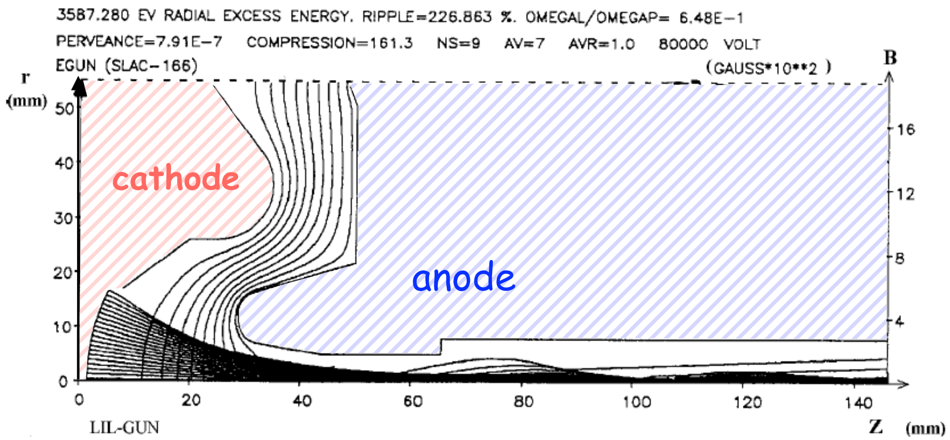
This force can be used to accelerate and decelerate particles

ACCELERATION BY TIME-VARYING FIELDS (2/10)

Electrostatic accelerators



- The potential difference between two electrodes is used to accelerate particles
- Limited in energy by the maximum high voltage (~ 10 MV)
- Present applications: x-ray tubes, low energy ions, electron sources (thermionic guns)



Electric field potential and beam trajectories inside an electron gun (LEP Injector Linac at CERN), computed with the code E-GUN

Comparison of magnetic and electric forces

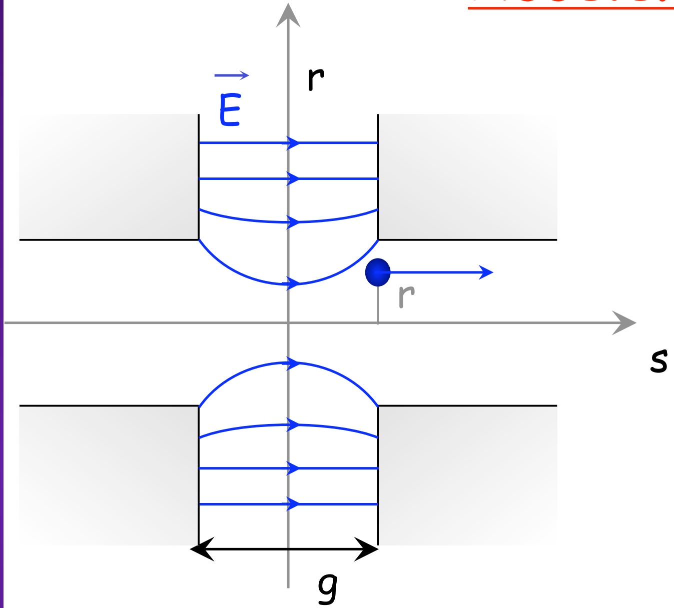
$$|\vec{B}| = 1 \text{ T}$$

$$|\vec{E}| = 10 \text{ MV/m}$$

$$\frac{F_{MAGN}}{F_{ELEC}} = \frac{e\,v\,B}{e\,E} = \beta\,c\,\frac{B}{E} \cong 3 \cdot 10^8 \,\frac{1}{10^7}\,\beta = 30\;\beta$$

ACCELERATION BY TIME-VARYING FIELDS (4/10)

Acceleration by time-varying electric field



- Let V_{RF} be the amplitude of the RF voltage across the gap g
- The particle crosses the gap at a distance r
- The energy gain is:

$$\Delta E = e \int_{-g/2}^{g/2} \vec{E}(s,r,t)\, \mathrm{d}\vec{s}$$

[MeV] [n] (1 for electrons or protons) [MV/m]

In the cavity gap, the electric field is supposed to be:

$$E(s,r,t) = E_1(s,r) \cdot E_2(t)$$

In general, $E_2(t)$ is a sinusoidal time variation with angular frequency ω_{RF}

$$E_2(t) = E_\circ \sin \Phi(t) \qquad \text{where} \qquad \Phi(t) = \int_{t_0}^{t} \omega_{RF} \mathrm{d}t + \Phi_0$$

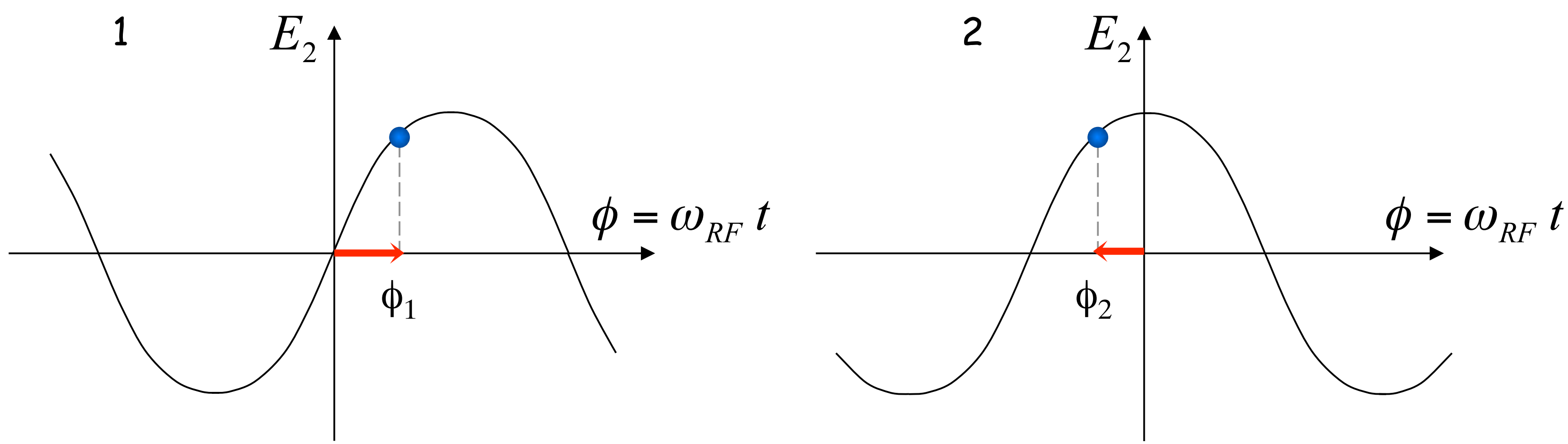
ACCELERATION BY TIME-VARYING FIELDS (5/10)

Convention

1. For circular accelerators, the origin of time is taken at the zero crossing of the RF voltage with positive slope

2. For linear accelerators, the origin of time is taken at the positive crest of the RF voltage

Time t= 0 chosen such that:



$$E_2(t) = E_\circ \sin\left(\omega_{RF}\, t\right) \qquad E_2(t) = E_\circ \cos\left(\omega_{RF}\, t\right)$$

ACCELERATION BY TIME-VARYING FIELDS (6/10)

First derivatives

$$\mathrm{d}\beta = \beta^{-1}\,\gamma^{-3}\,\mathrm{d}\gamma$$

$$\mathrm{d}(cp) = E_0\,\gamma^3\,\mathrm{d}\beta$$

$$\mathrm{d}\gamma = \beta\left(1-\beta^2\right)^{-3/2}\mathrm{d}\beta$$

Logarithmic derivatives

$$\frac{\mathrm{d}\beta}{\beta} = (\beta\,\gamma)^{-2}\,\frac{\mathrm{d}\gamma}{\gamma}$$

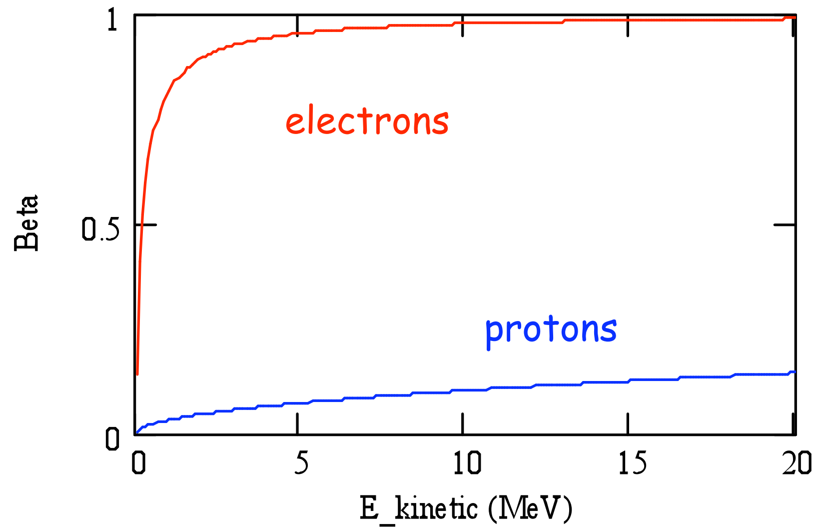
$$\frac{\mathrm{d}p}{p} = \frac{\gamma^2}{\gamma^2-1}\,\frac{\mathrm{d}E}{E} = \frac{\gamma}{\gamma+1}\,\frac{\mathrm{d}E_{kin}}{E_{kin}}$$

$$\frac{\mathrm{d}\gamma}{\gamma} = \left(\gamma^2-1\right)\frac{\mathrm{d}\beta}{\beta}$$

ACCELERATION BY TIME-VARYING FIELDS (7/10)

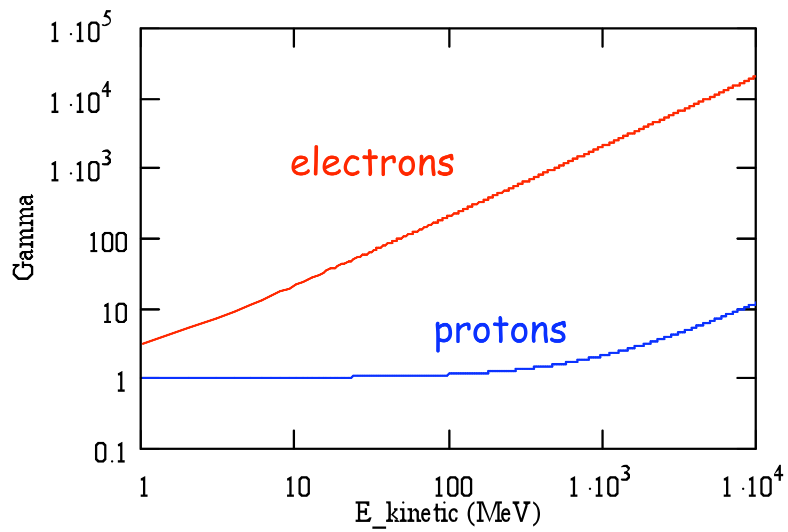
normalized velocity

$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}$$



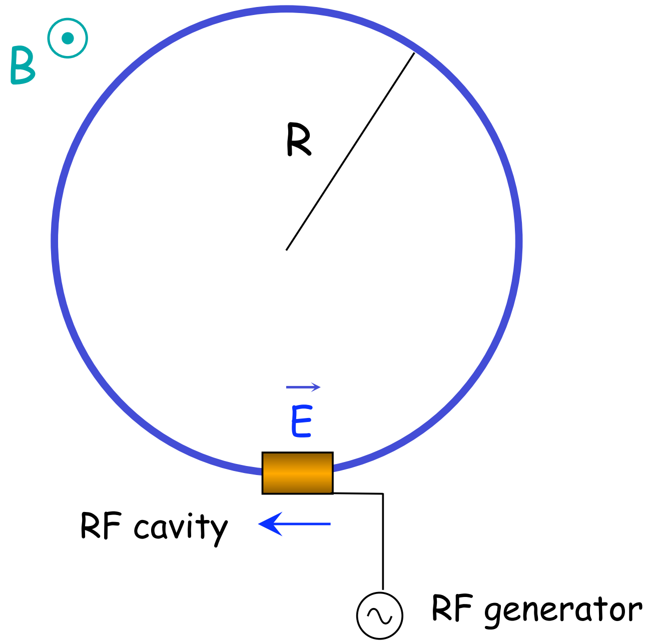
total energy / rest energy

$$\gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}}$$



ACCELERATION BY TIME-VARYING FIELDS (8/10)

Synchrotron



Synchronism condition

$$T_s = h\,T_{RF}$$

$$\frac{2\pi\,R}{v_s} = h\,T_{RF}$$

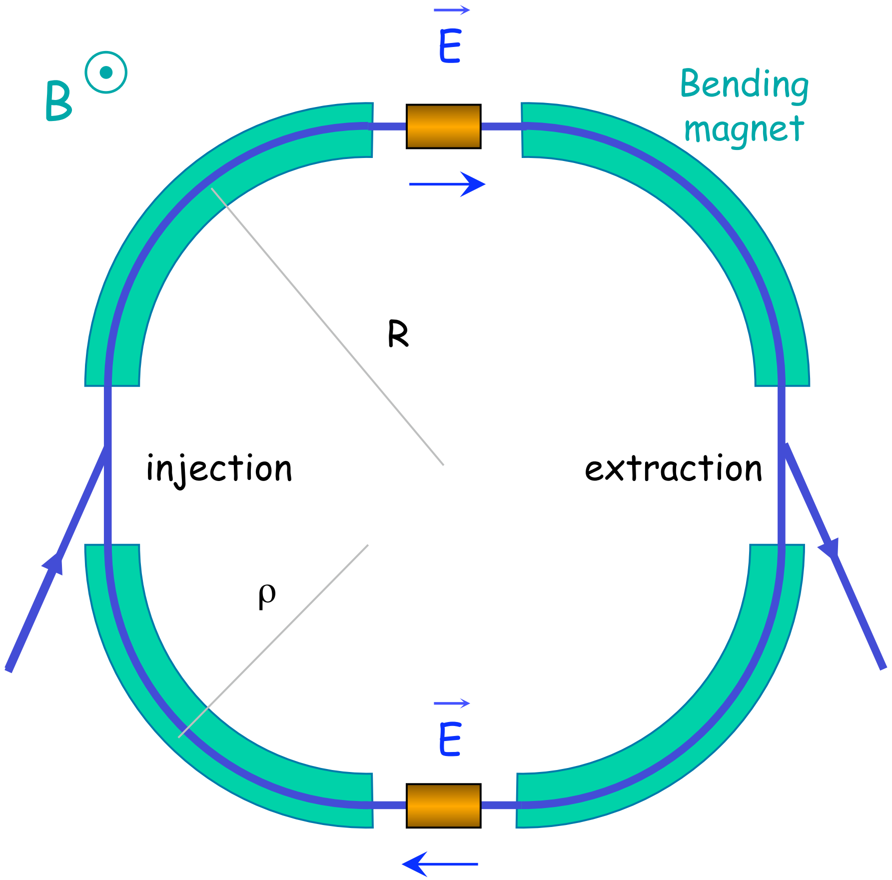
h integer, harmonic number

1. ω_{RF} and ω increase with energy

2. To keep particles on the closed orbit, B should increase with time

ACCELERATION BY TIME-VARYING FIELDS (9/10)

Synchrotron



- In reality, the orbit in a synchrotron is not a circle, straight sections are added for RF cavities, injection and extraction, etc..
- Usually the beam is pre-accelerated in a linac (or a smaller synchrotron) before injection
- The bending radius ρ does not coincide to the machine radius $R = L/2\pi$

ACCELERATION BY TIME-VARYING FIELDS (10/10)

Parameters for circular accelerators

The basic principles, for the common circular accelerators, are based on the two relations:

1. The Lorentz equation: the orbit radius can be espressed as:

$$R = \frac{\gamma \, v \, m_0}{eB}$$

2. The synchronicity condition: The revolution frequency can be expressed as:

$$f = \frac{e\,B}{2\pi \, \gamma \, m_0}$$

According to the parameter we want to keep constant or let vary, one has different acceleration principles. They are summarized in the table below:

Machine	Energy (γ)	Velocity	Field	Orbit	Frequency
Cyclotron	~ 1	var.	const.	~ v	const.
Synchrocyclotron	var.	var.	B(r)	~ p	B(r)/γ(t)
Proton/Ion synchrotron	var.	var.	~ p	R	~ v
Electron synchrotron	var.	const.	~ p	R	const.

TRANSIT TIME FACTOR (1/2)

Transit time factor

RF acceleration in a gap **g**

$$E(s,r,t) = E_1(s,r) \cdot E_2(t)$$

Simplified model ➡

$$E_1(s,r) = \frac{V_{RF}}{g} = \text{const.}$$

$$E_2(t) = \sin(\omega_{RF}\, t + \phi_0)$$

At t = 0, s = 0 and v ≠ 0, parallel to the electric field

Energy gain:

$$\Delta E = e \int_{-g/2}^{g/2} E(s,r,t)\, \mathrm{d}s \quad \Rightarrow \quad \Delta E = e\, V_{RF}\, T_a \sin\phi_0$$

where

$$T_a = \frac{\sin \dfrac{\omega_{RF}\, g}{2\, v}}{\dfrac{\omega_{RF}\, g}{2\, v}}$$

T_a is called transit time factor

- $T_a < 1$
- $T_a \rightarrow 1$ if $g \rightarrow 0$

TRANSIT TIME FACTOR (2/2)

Transit time factor II

In the general case, the transit time factor is given by:

$$T_a = \frac{\int_{-\infty}^{+\infty} E_1(s,r)\,\cos\left(\omega_{RF}\,\frac{s}{v}\right)\,\mathrm{d}s}{\int_{-\infty}^{+\infty} E_1(s,r)\,\mathrm{d}s}$$

It is the ratio of the peak energy gained by a particle with velocity **v** to the peak energy gained by a particle with infinite velocity.

MAIN RF PARAMETERS

Main RF parameters

I. Voltage, phase, frequency

In order to accelerate particles, longitudinal fields must be generated in the direction of the desired acceleration

$$E(s,t) = E_1(s) \cdot E_2(t) \qquad E_2(t) = E_0 \sin\left[\int_{t_0}^{t} \omega_{RF}\, \mathrm{d}t + \phi_0\right]$$

$$\omega_{RF} = 2\pi\, f_{RF} \qquad \Delta E = e\, V_{RF}\, T_a\, \sin\phi_0$$

Such electric fields are generated in RF cavities characterized by the voltage amplitude, the frequency and the phase

II. Harmonic number

$$T_{rev} = h\, T_{RF} \quad \Rightarrow \quad f_{RF} = h\, f_{rev}$$

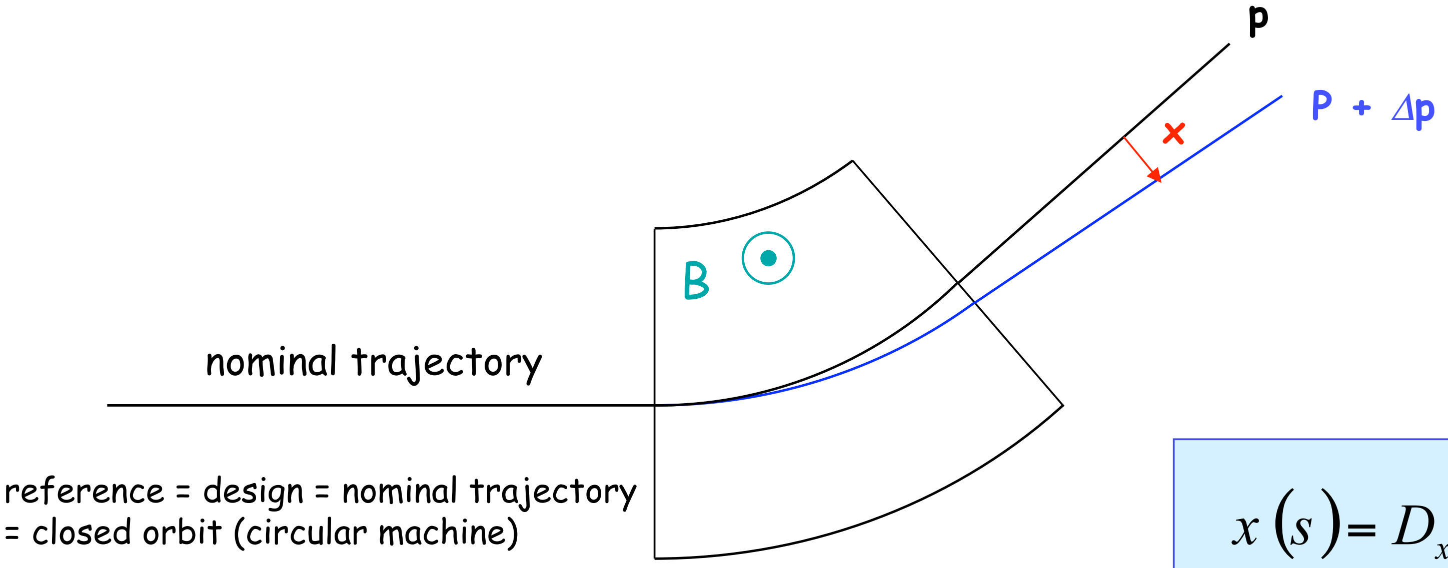
f_{rev} = revolution frequency
f_{RF} = frequency of the RF
h = harmonic number

harmonic number in different machines:

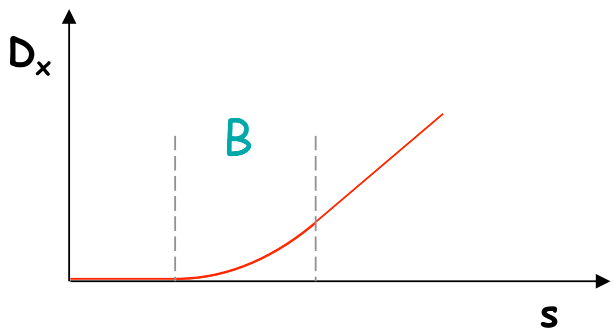
AA	EPA	PS	SPS
1	8	20	4620

MOMENTUM COMPACTION FACTOR (1/6)

Dispersion



$$x(s) = D_x(s)\,\frac{\Delta p}{p}$$



MOMENTUM COMPACTION FACTOR (2/6)

Momentum compaction factor in a transport system

In a particle transport system, a nominal trajectory is defined for the nominal momentum **p**.

For a particle with a momentum **p + Δp** the trajectory length can be different from the length **L** of the nominal trajectory.

The momentum compaction factor is defined by the ratio:

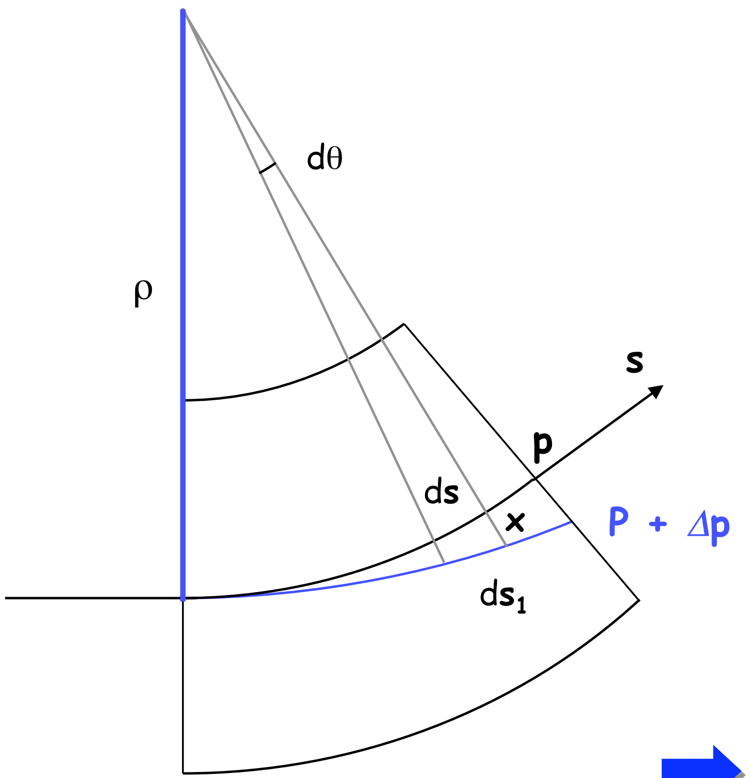
$$\alpha_p = \frac{dL/L}{dp/p}$$

Therefore, for small momentum deviation, to first order it is:

$$\frac{\Delta L}{L} = \alpha_p \frac{\Delta p}{p}$$

MOMENTUM COMPACTION FACTOR (3/6)

Example: constant magnetic field



$$ds = \rho \, \mathrm{d}\theta$$

$$ds_1 = (\rho + x)\mathrm{d}\theta$$

$$\frac{ds_1 - ds}{ds} = \frac{(\rho + x)\mathrm{d}\theta - \rho \, \mathrm{d}\theta}{\rho \, \mathrm{d}\theta} = \frac{x}{\rho} = \frac{D_x}{\rho} \frac{\mathrm{d}p}{p}$$

By definition of dispersion D_x

$$\alpha_p = \frac{1}{L} \int_0^L \frac{D_x(s)}{\rho(s)} \mathrm{d}s$$

To first order, only the bending magnets contribute to a change of the trajectory length (r = ∞ in the straight sections)

MOMENTUM COMPACTION FACTOR (4/6)

Momentum compaction in a ring

In a circular accelerator, a nominal closed orbit is defined for the nominal momentum **p**.

For a particle with a momentum deviation Δ**p** produces an orbit length variation Δ**C** with:

For **B** = const.

$$\frac{\Delta C}{C} = \alpha_p \frac{\Delta p}{p}$$

$$C = 2\pi R$$

circumference — (average) radius of the closed orbit

The momentum compaction factor is defined by the ratio:

$$\alpha_p = \frac{dC/C}{dp/p} = \frac{dR/R}{dp/p}$$

and

$$\alpha_p = \frac{1}{C}\int_C \frac{D_x(s)}{\rho(s)}\,\mathrm{d}s$$

N.B.: in most circular machines, α_p is positive $\Rightarrow$ higher momentum means longer circumference

MOMENTUM COMPACTION FACTOR (5/6)

Momentum compaction as a function of energy

$$E = \frac{p\,c}{\beta} \quad \Rightarrow \quad \frac{\mathrm{d}E}{E} = \beta^2 \frac{dp}{p}$$

$$\alpha_p = \beta^2 \frac{E}{R} \frac{\mathrm{d}R}{\mathrm{d}E}$$

MOMENTUM COMPACTION FACTOR (6/6)

Momentum compaction as a function of magnetic field

Definition of average magnetic field

$$< B > = \frac{1}{2\pi R} \int_C B_f \, \mathrm{d}s = \frac{1}{2\pi R} \left(\int_{straights} B_f \, \mathrm{d}s + \int_{magnets} B_f \, \mathrm{d}s \right)$$

$$\int_{straights} B_f \, \mathrm{d}s = 0 \qquad \int_{magnets} B_f \, \mathrm{d}s = 2\pi \rho B_f$$

$$< B > = \frac{B_f \, \rho}{R} \quad \Rightarrow \quad \frac{\mathrm{d} < B >}{< B >} = \frac{\mathrm{d} B_f}{B_f} + \frac{\mathrm{d}\rho}{\rho} - \frac{\mathrm{d}R}{R}$$

$$B_f \, \rho = \frac{p}{e}$$

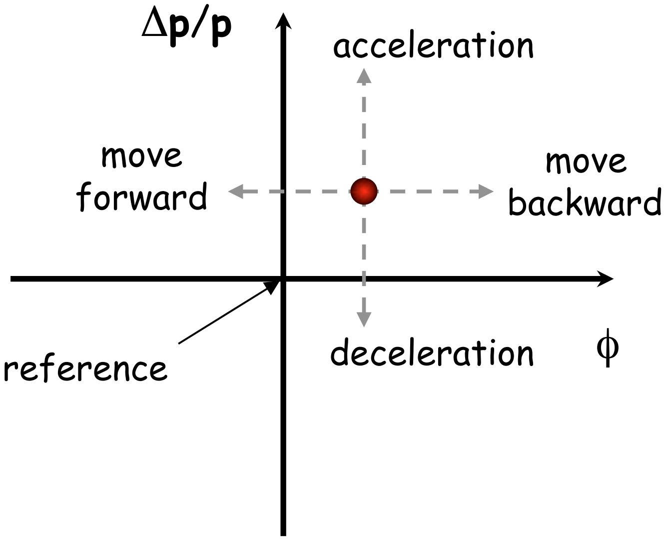
$$< B > R = \frac{p}{e} \quad \Rightarrow \quad \frac{\mathrm{d} < B >}{< B >} + \frac{\mathrm{d}R}{R} = \frac{\mathrm{d}p}{p}$$

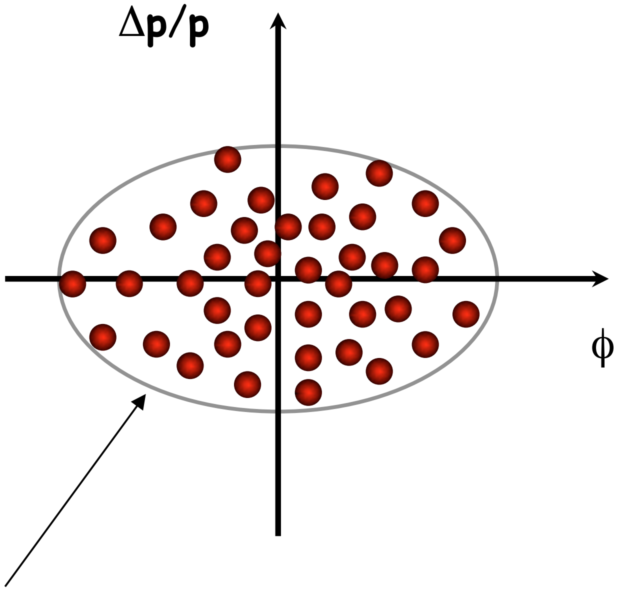
For B_f = const.

$$\alpha_p = 1 - \frac{\mathrm{d} < B >}{< B >} \Bigg/ \frac{\mathrm{d}p}{p}$$

LONGITUDINAL PHASE SPACE (1/2)

Longitudinal phase space





The particle trajectory in the phase space (Δ**p**/**p**, ϕ) describes its longitudinal motion.

Emittance: phase space area including all the particles

NB: if the emittance contour correspond to a possible orbit in phase space, its shape does not change with time (matched beam)

LONGITUDINAL PHASE SPACE (2/2)

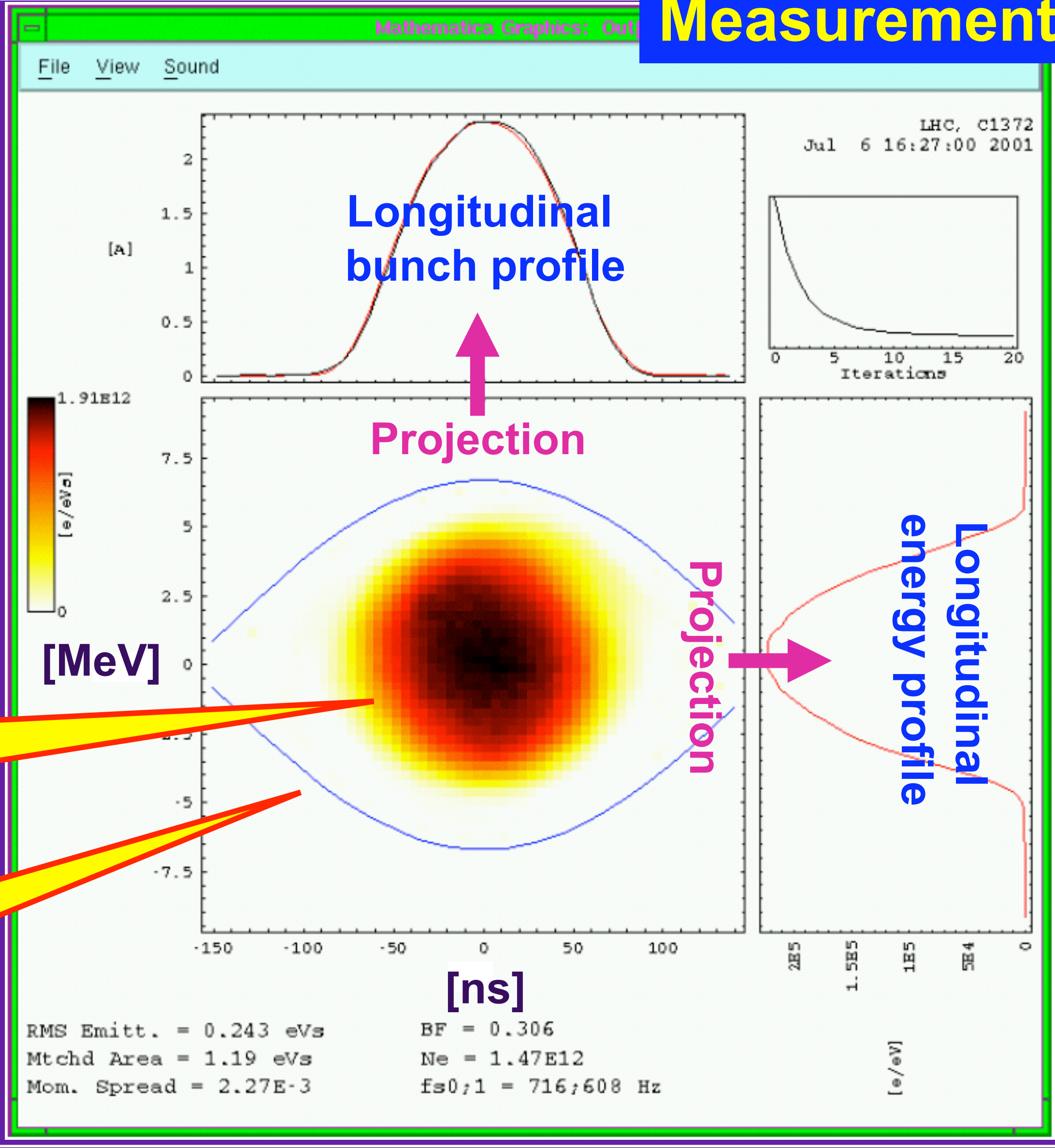
TOMOSCOPE (developed by S. Hancock, CERN/AB/RF)

The aim of TOMOGRAPHY is to estimate an unknown distribution (here the 2D longitudinal distribution) using only the information in the bunch profiles

Surface = Longitudinal EMITTANCE of the bunch = ε_L [eV.s]

Surface = Longitudinal ACCEPTANCE of the bucket

Measurement



TRANSITION ENERGY (1/3)

Transition energy

Proton (ion) circular machine with α_p positive

1. Momentum larger than the nominal ($p + \Delta p$) $\Rightarrow$ longer orbit ($C+\Delta C$)
2. Momentum larger than the nominal ($p + \Delta p$) $\Rightarrow$ higher velocity ($v + \Delta v$)

What happens to the revolution frequency $f = v/C$?

- At low energy, v increases faster than C with momentum
- At high energy $v \cong c$ and remains almost constant

➡ There is an energy for which the velocity variation is compensated by the trajectory variation $\Rightarrow$ transition energy

Below transition:	higher energy	$\Rightarrow$	higher revolution frequency
Above transition:	higher energy	$\Rightarrow$	lower revolution frequency

TRANSITION ENERGY (2/3)

Transition energy – quantitative approach

We define a parameter η (revolution frequency spread per unit of momentum spread):

$$\eta = \frac{\mathrm{d}f / f}{\mathrm{d}p / p} = \frac{\mathrm{d}\omega / \omega}{\mathrm{d}p / p}$$

$$f = \frac{v}{C} \quad \Rightarrow \quad \frac{\mathrm{d}f}{f} = \frac{\mathrm{d}\beta}{\beta} - \frac{\mathrm{d}C}{C}$$

from $p = \frac{m_0 \, c \, \beta}{\sqrt{1-\beta^2}}$ $\Rightarrow$ $\frac{\mathrm{d}\beta}{\beta} = \frac{1}{\gamma^2} \frac{\mathrm{d}p}{p}$

definition of momentum compaction factor: $\frac{\mathrm{d}C}{C} = \alpha_p \frac{\mathrm{d}p}{p}$

$$\frac{\mathrm{d}f}{f} = \left(\frac{1}{\gamma^2} - \alpha_p \right) \frac{\mathrm{d}p}{p}$$

TRANSITION ENERGY (3/3)

Transition energy – quantitative approach

$$\eta = \frac{1}{\gamma^2} - \alpha_p$$

The transition energy is the energy that corresponds to $\eta = 0$

(α_p is fixed, and γ variable)

$$\gamma_{tr} = \sqrt{\frac{1}{\alpha_p}}$$

The parameter η can also be written as

$$\eta = \frac{1}{\gamma^2} - \frac{1}{\gamma_{tr}^2}$$

- At low energy $\eta > 0$
- At high energy $\eta < 0$

N.B.: for electrons, $\gamma \gg \gamma_{tr} \Rightarrow \eta < 0$

for linacs $\alpha_p = 0 \Rightarrow \eta > 0$

4 EQUATIONS RELATED TO SYNCHROTRONS (1/6)

Equations related to synchrotrons

$$\frac{\mathrm{d}p}{p} = \gamma_{tr}^{2} \frac{\mathrm{d}R}{R} + \frac{\mathrm{d}B}{B}$$

$$\frac{\mathrm{d}p}{p} = \gamma^{2} \frac{\mathrm{d}f}{f} + \gamma^{2} \frac{\mathrm{d}R}{R}$$

$$\frac{\mathrm{d}B}{B} = \gamma_{tr}^{2} \frac{\mathrm{d}f}{f} + \left[1 - \left(\frac{\gamma_{tr}}{\gamma}\right)^{2}\right] \frac{\mathrm{d}p}{p}$$

$$\frac{\mathrm{d}B}{B} = \gamma^{2} \frac{\mathrm{d}f}{f} + \left(\gamma^{2} - \gamma_{tr}^{2}\right) \frac{\mathrm{d}R}{R}$$

$p\ [\mathrm{MeV/c}]$	momentum
$R\ [\mathrm{m}]$	orbit radius
$B\ [\mathrm{T}]$	magnetic field
$f\ [\mathrm{Hz}]$	rev. frequency
γ_{tr}	transition energy

4 EQUATIONS RELATED TO SYNCHROTRONS (2/6)

I - Constant radius

$$\mathrm{d}R = 0$$

Beam maintained on the same orbit when energy varies

$$\frac{\mathrm{d}p}{p} = \frac{\mathrm{d}B}{B}$$

$$\frac{\mathrm{d}p}{p} = \gamma^2 \frac{\mathrm{d}f}{f}$$

If **p** increases

$\Rightarrow$ **B** increases

f increases

4 EQUATIONS RELATED TO SYNCHROTRONS (3/6)

II - Constant energy

$$\mathrm{d}p = 0$$

$$V_{RF} = 0$$

Beam debunches

$$\frac{\mathrm{d}p}{p} = 0 = \gamma_{tr}^{2} \frac{\mathrm{d}R}{R} + \frac{\mathrm{d}B}{B}$$

$$\frac{\mathrm{d}p}{p} = 0 = \gamma^{2} \frac{\mathrm{d}f}{f} + \gamma^{2} \frac{\mathrm{d}R}{R}$$

If B increases

→ R decreases

f increases

4 EQUATIONS RELATED TO SYNCHROTRONS (4/6)

III - Magnetic flat-top

$$\mathrm{d}B = 0$$

Beam bunched with constant magnetic field

$$\frac{\mathrm{d}p}{p} = \gamma_{tr}^{\;2} \frac{\mathrm{d}R}{R}$$

$$\frac{\mathrm{d}B}{B} = 0 = \gamma_{tr}^{\;2} \frac{\mathrm{d}f}{f} + \left[1 - \left(\frac{\gamma_{tr}}{\gamma}\right)^2\right] \frac{\mathrm{d}p}{p}$$

$$\frac{\mathrm{d}B}{B} = 0 = \gamma^2 \frac{\mathrm{d}f}{f} + \left(\gamma^2 - \gamma_{tr}^2\right) \frac{\mathrm{d}R}{R}$$

If **p** increases

➡ **R** increases

f increase $\gamma < \gamma_{tr}$

decreases $\gamma > \gamma_{tr}$

4 EQUATIONS RELATED TO SYNCHROTRONS (5/6)

IV – Constant frequency

$$\mathrm{d}f = 0$$

Beam driven by an external oscillator

$$\frac{\mathrm{d}p}{p} = \gamma^2 \frac{\mathrm{d}R}{R}$$

$$\frac{\mathrm{d}B}{B} = \left[1 - \left(\frac{\gamma_{tr}}{\gamma}\right)^2\right] \frac{\mathrm{d}p}{p}$$

$$\frac{\mathrm{d}B}{B} = \left(\gamma^2 - \gamma_{tr}^2\right) \frac{\mathrm{d}R}{R}$$

If p increases

→ R increases

B decreases $\gamma < \gamma_{tr}$

increase $\gamma > \gamma_{tr}$

4 EQUATIONS RELATED TO SYNCHROTRONS (6/6)

Four conditions - resume

Beam	Parameter	Variations
Debunched	$\Delta p = 0$	$B \Uparrow, R \Downarrow, f \Uparrow$
Fixed orbit	$\Delta R = 0$	$B \Uparrow, p \Uparrow, f \Uparrow$
Magnetic flat-top	$\Delta B = 0$	$p \Uparrow, R \Uparrow, f \Uparrow (\eta > 0)$ $f \Downarrow (\eta < 0)$
External oscillator	$\Delta f = 0$	$B \Uparrow, p \Downarrow, R \Downarrow (\eta > 0)$ $p \Uparrow, R \Uparrow (\eta < 0)$

p momentum

R orbit radius

B magnetic field

f frequency

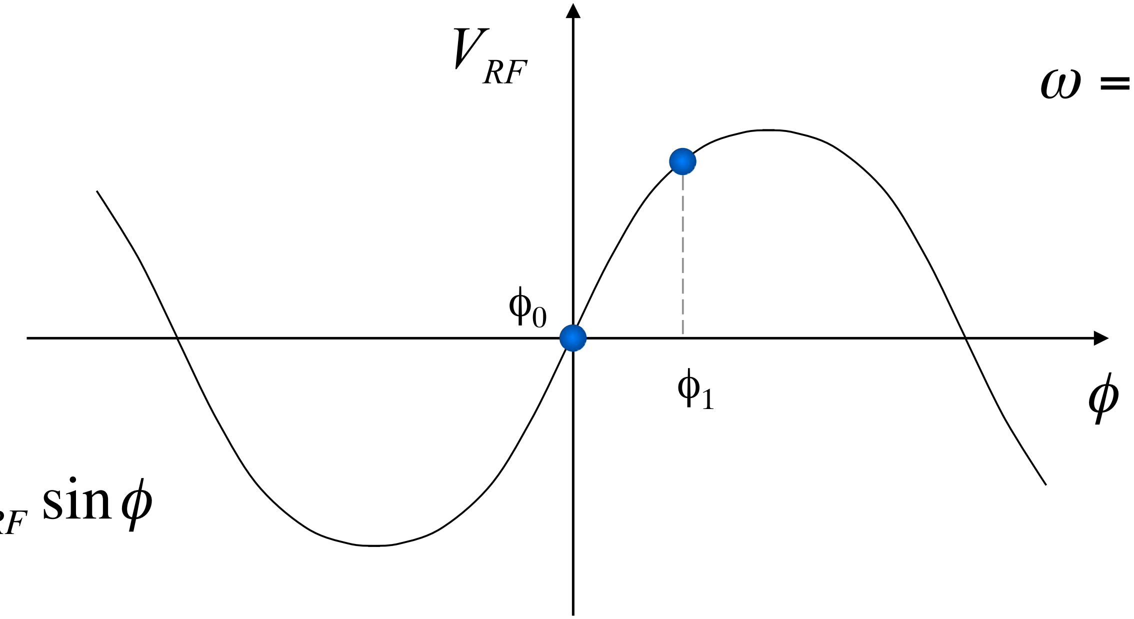
SYNCHROTRON OSCILLATIONS AND PHASE STABILITY (1/6)

Synchronous particle

Simple case (no accel.): B = const. $\gamma < \gamma_{tr}$

Synchronous particle: particle that sees always the same phase (at each turn) in the RF cavity

$$\omega = \frac{e\,B}{\gamma\, m_0} = \frac{\omega_{RF}}{h}$$



$$\Delta E = e\,\hat{V}_{RF} \sin\phi$$

In order to keep the resonant condition, the particle must keep a constant energy

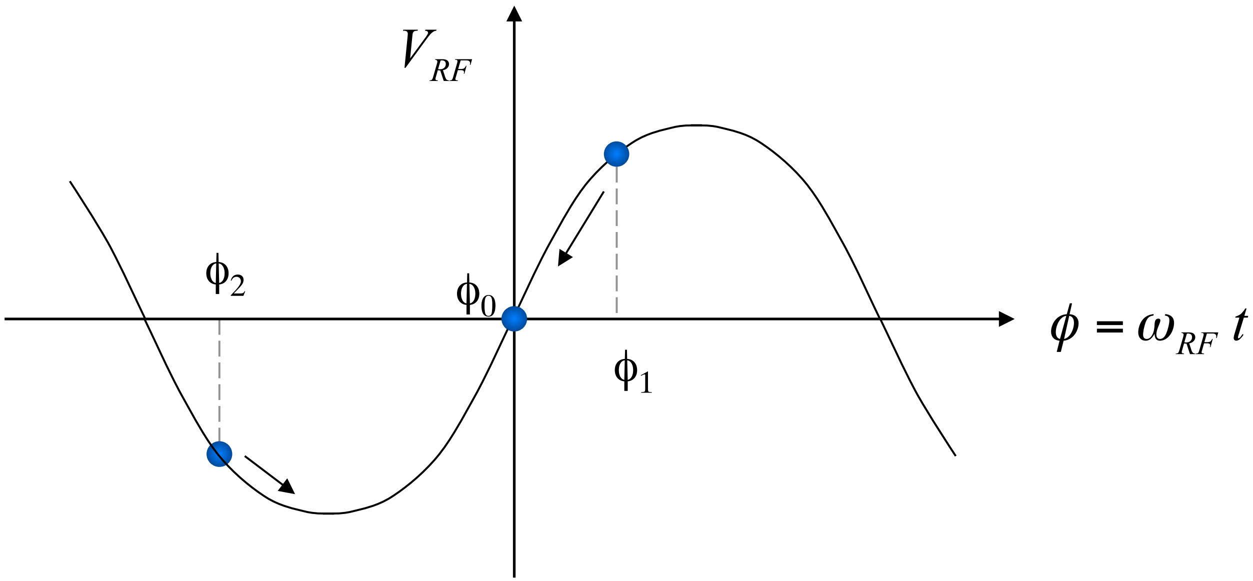
The phase of the synchronous particle must therefore be $\phi_0 = 0$ (circular machines convention)

Let's see what happens for a particle with the same energy and a different phase (e.g., ϕ_1)

SYNCHROTRON OSCILLATIONS AND PHASE STABILITY (2/6)

Synchrotron oscillations

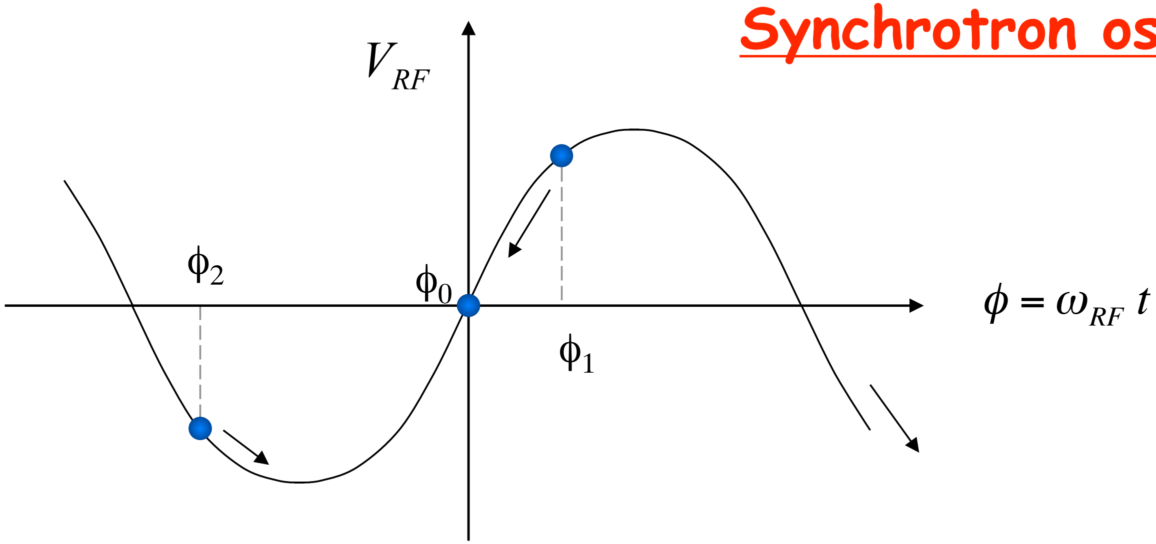
ϕ_1 - The particle is accelerated
- Below transition, an increase in energy means an increase in revolution frequency
- The particle arrives earlier – tends toward ϕ_0



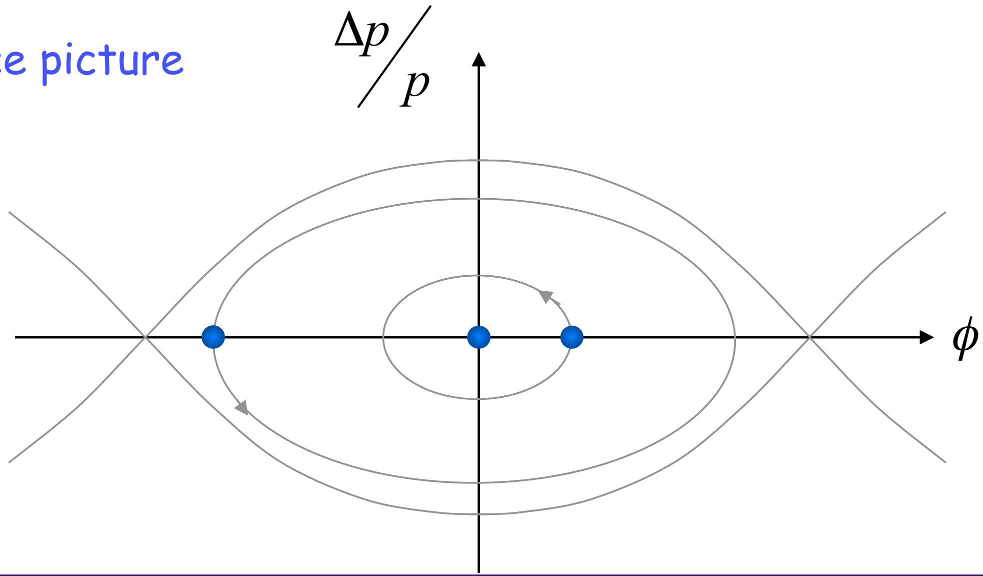
ϕ_2 - The particle is decelerated
- decrease in energy - decrease in revolution frequency
- The particle arrives later – tends toward ϕ_0

SYNCHROTRON OSCILLATIONS AND PHASE STABILITY (3/6)

Synchrotron oscillations



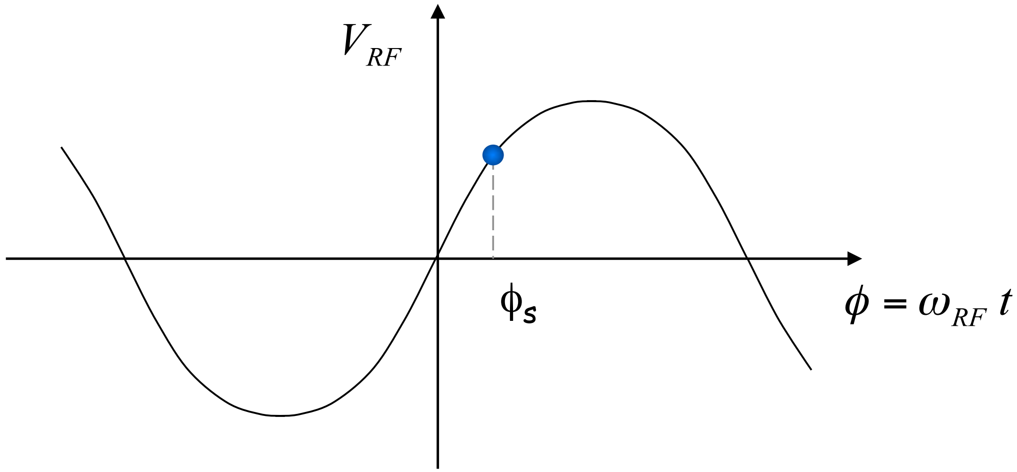
Phase space picture



SYNCHROTRON OSCILLATIONS AND PHASE STABILITY (4/6)

Synchronous particle

Case with acceleration B increasing $\gamma < \gamma_{tr}$



$$\Delta E = e\hat{V}_{RF} \sin\phi$$

The phase of the synchronous particle is now $\phi_s > 0$ (circular machines convention)

The synchronous particle accelerates, and the magnetic field is increased accordingly to keep the constant radius R

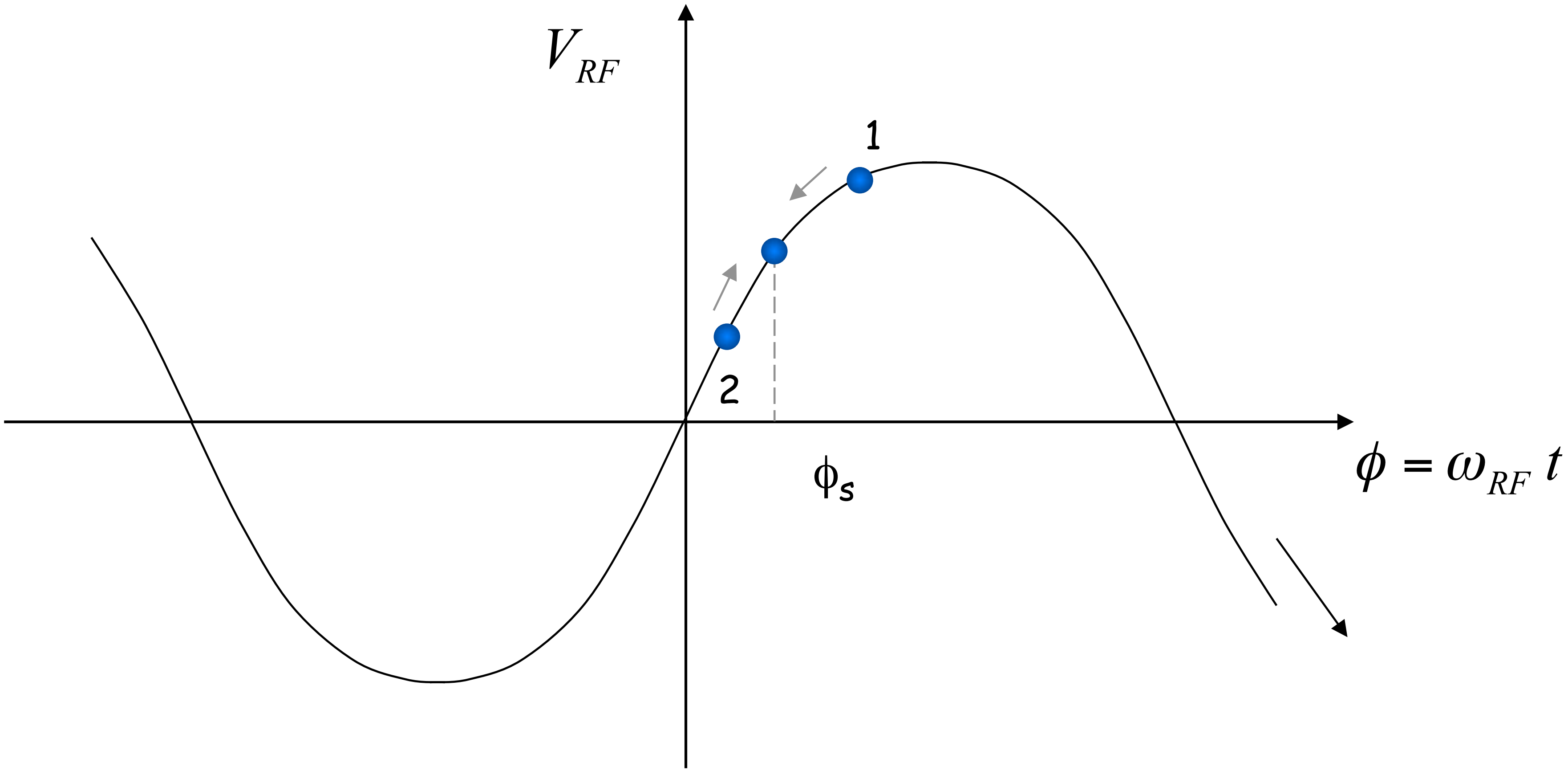
$$R = \frac{\gamma\, v\, m_0}{eB}$$

The RF frequency is increased as well in order to keep the resonant condition

$$\omega = \frac{e\,B}{\gamma\, m_0} = \frac{\omega_{RF}}{h}$$

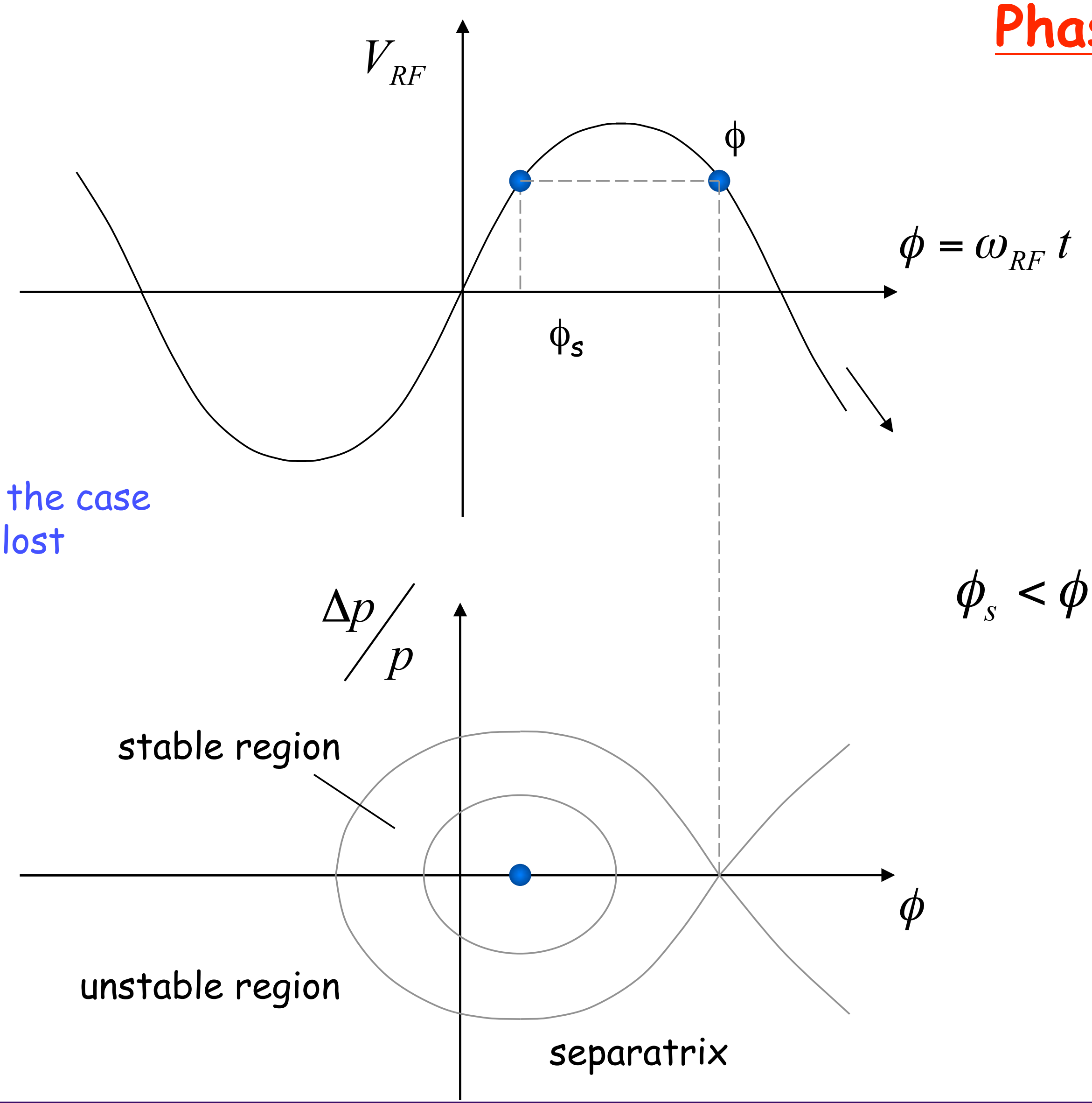
SYNCHROTRON OSCILLATIONS AND PHASE STABILITY (5/6)

Phase stability



SYNCHROTRON OSCILLATIONS AND PHASE STABILITY (6/6)

Phase stability



The symmetry of the case with B = const. is lost

$$\phi_s < \phi < \pi - \phi_s$$

EQUATIONS OF MOTION (1/11)

RF acceleration for synchronous particle - energy gain

Let's assume a synchronous particle with a given $\phi_s > 0$

We want to calculate its rate of acceleration, and the related rate of increase of B, f.

$$p = e\,B\,\rho$$

Want to keep ρ = const

$$\Rightarrow \quad \frac{\mathrm{d}p}{\mathrm{d}t} = e\,\rho\,\frac{\mathrm{d}\,B}{\mathrm{d}t} = e\,\rho\,\dot{B}$$

Over one turn: $$(\Delta\,p)_{turn} = e\,\rho\,\dot{B}\,T_{rev} = e\,\rho\,\dot{B}\,\frac{2\pi\,R}{\beta\,c}$$

We know that (relativistic equations) : $$\Delta\,p = \frac{\Delta E}{\beta\,c}$$

$$\Rightarrow \quad \boxed{(\Delta E)_{turn} = e\,\rho\,\dot{B}\;2\pi\,R}$$

EQUATIONS OF MOTION (2/11)

RF acceleration for synchronous particle - phase

$$(\Delta E)_{turn} = e\,\rho\,\dot{B}\;2\pi\,R$$

On the other hand, for the synchronous particle:

$$(\Delta E)_{turn} = e\,\hat{V}_{RF}\,\sin\phi_s$$

$$e\,\rho\,\dot{B}\;2\pi\,R = e\,\hat{V}_{RF}\,\sin\phi_s$$

Therefore:

1. Knowing ϕ_s, one can calculate the increase rate of the magnetic field needed for a given RF voltage:

$$\dot{B} = \frac{\hat{V}_{RF}}{2\pi\,\rho\,R}\sin\phi_s$$

2. Knowing the magnetic field variation and the RF voltage, one can calculate the value of the synchronous phase:

$$\sin\phi_s = 2\pi\,\rho\,R\,\frac{\dot{B}}{\hat{V}_{RF}} \quad \Rightarrow \quad \phi_s = \arcsin\left(2\pi\,\rho\,R\,\frac{\dot{B}}{\hat{V}_{RF}}\right)$$

EQUATIONS OF MOTION (3/11)

RF acceleration for synchronous particle - frequency

$$\omega_{RF} = h\omega_s = h\frac{e}{m} < B > \qquad \left(v = \frac{e}{m} B\rho \right)$$

$$\omega_{RF} = h\frac{e}{m}\frac{\rho}{R} B$$

From relativistic equations:

$$\omega_{RF} = \frac{hc}{R}\sqrt{\frac{B^2}{B^2 + \left(E_0 / ec\rho\right)^2}}$$

Let

$$B_0 \equiv \frac{E_0}{ec\rho} \Rightarrow f_{RF} = \frac{hc}{2\pi R}\left(\frac{B}{B_0}\right)\frac{1}{\sqrt{1+\left(B/B_0\right)^2}}$$

EQUATIONS OF MOTION (4/11)

RF acceleration for non synchronous particle

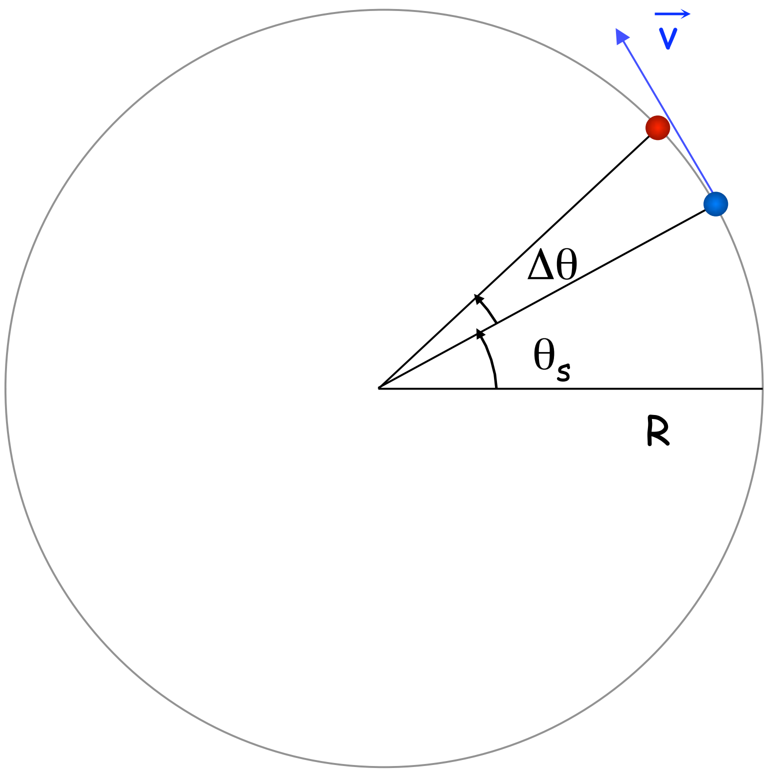
Parameter definition (subscript “s” stands for synchronous particle):

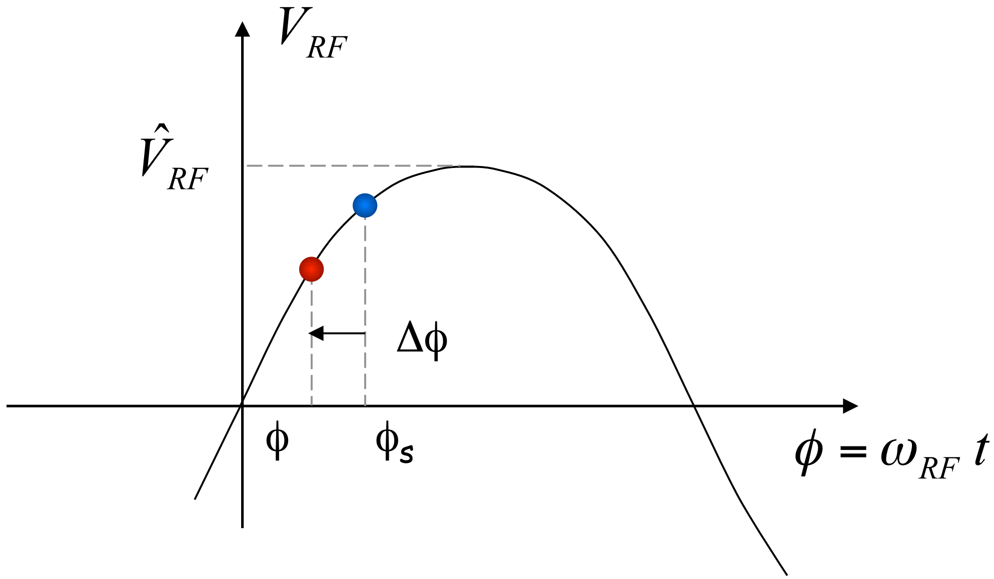
$f = f_s + \Delta f$	revolution frequency
$\phi = \phi_s + \Delta\phi$	RF phase
$p = p_s + \Delta p$	Momentum
$E = E_s + \Delta E$	Energy
$\theta = \theta_s + \Delta\theta$	Azimuth angle

$$\mathrm{d}s = R\,\mathrm{d}\theta$$

$$\theta(t) = \int_{t_0}^{t} \omega(\tau)\,\mathrm{d}\tau$$

EQUATIONS OF MOTION (5/11)





$$\Delta\theta > 0 \quad \Rightarrow \quad \Delta\phi < 0$$

Since $f_{RF} = h\, f_{rev}$

$$\Delta\phi = -h\, \Delta\theta$$

Over one turn θ varies by 2π

ϕ varies by $2\pi h$

EQUATIONS OF MOTION (6/11)

Parameters versus $\dot{\phi}$

1. Angular frequency

$$\theta(t) = \int_{t_0}^{t} \omega(\tau)\,\mathrm{d}\tau$$

$$\Delta\omega = \frac{\mathrm{d}}{\mathrm{d}t}(\Delta\theta)$$

$$= -\frac{1}{h}\frac{\mathrm{d}}{\mathrm{d}t}(\Delta\phi)$$

$$= -\frac{1}{h}\frac{\mathrm{d}}{\mathrm{d}t}(\phi - \phi_s)$$

$\frac{\mathrm{d}\phi_s}{\mathrm{d}t} = 0$ by definition

$$= -\frac{1}{h}\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

→ $$\Delta\omega = -\frac{1}{h}\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

EQUATIONS OF MOTION (7/11)

Parameters versus $\dot{\phi}$

2. Momentum

$$\eta = \frac{\mathrm{d}\omega / \omega}{\mathrm{d}p / p} = \frac{\Delta\omega / \omega}{\Delta p / p}$$

$$\Delta p = \frac{p_s}{\omega_s} \frac{\Delta\omega}{\eta} = \frac{p_s}{\omega_s \eta} \left(- \frac{1}{h} \frac{\mathrm{d}\phi}{\mathrm{d}t} \right)$$

$$\Rightarrow \quad \Delta p = \frac{- p_s}{\omega_s \eta h} \frac{\mathrm{d}\phi}{\mathrm{d}t}$$

3. Energy

$$\frac{\mathrm{d}E}{\mathrm{d}p} = v$$

$$\frac{\Delta E}{\Delta p} = v = \omega R$$

$$\Rightarrow \quad \Delta E = - \frac{R \, p_s}{\eta \, h} \frac{\mathrm{d}\phi}{\mathrm{d}t}$$

EQUATIONS OF MOTION (8/11)

Derivation of equations of motion

Energy gain after the RF cavity

$$(\Delta E)_{turn} = e\hat{V}_{RF} \sin\phi$$

$$(\Delta p)_{turn} = \frac{e}{\omega R} \hat{V}_{RF} \sin\phi$$

Average increase per time unit

$$\frac{(\Delta p)_{turn}}{T_{rev}} = \frac{e}{2\pi R} \hat{V}_{RF} \sin\phi \qquad 2\pi R \dot{p} = e \hat{V}_{RF} \sin\phi$$

valid for any particle !

$$\Rightarrow \quad 2\pi (R \dot{p} - R_s \dot{p}_s) = e \hat{V}_{RF} (\sin\phi - \sin\phi_s)$$

EQUATIONS OF MOTION (9/11)

Derivation of equations of motion

After some development (see J. Le Duff, in Proceedings CAS 1992, CERN 94-01)

$$2\pi \frac{d}{dt}\left(\frac{\Delta E}{\omega_s}\right) = e\,\hat{V}_{RF}\left(\sin\phi - \sin\phi_s\right)$$

An approximated version of the above is

$$\frac{\mathrm{d}(\Delta p)}{\mathrm{d}t} = \frac{e\,\hat{V}_{RF}}{2\pi\,R_s}\left(\sin\phi - \sin\phi_s\right)$$

Which, together with the previously found equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{\omega_s \eta\, h}{p_s}\Delta p$$

Describes the motion of the non-synchronous particle in the longitudinal phase space ($\Delta p, \phi$)

Equations of motion I

$$\begin{cases} \dfrac{\mathrm{d}(\Delta p)}{\mathrm{d}t} = A\,(\sin\phi - \sin\phi_s) \\ \dfrac{\mathrm{d}\phi}{\mathrm{d}t} = B\,\Delta p \end{cases}$$

with
$$A = \frac{e\,\hat{V}_{RF}}{2\pi\,R_s}$$

$$B = -\frac{\eta\,h}{p_s}\,\frac{\beta_s c}{R_s}$$

EQUATIONS OF MOTION (11/11)

Equations of motion II

1. First approximation – combining the two equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{B}\frac{\mathrm{d}\phi}{\mathrm{d}t}\right) - A\left(\sin\phi - \sin\phi_s\right) = 0$$

We assume that **A** and **B** <u>change very slowly</u> compared to the variable $\Delta\phi = \phi - \phi_s$

$$\Rightarrow \quad \frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} + \frac{\Omega_s^{\,2}}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

with $\frac{\Omega_s^{\,2}}{\cos\phi_s} = -A\,B$ We can also define: $\Omega_0^{\,2} = \frac{\Omega_s^{\,2}}{\cos\phi_s} = \frac{e\,\hat{V}_{RF}\,\eta\,h\,c^2}{2\pi\,R_s^{\,2}\,E_s}$

SMALL AND LARGE AMPLITUDE OSCILLATIONS (1/7)

Small amplitude oscillations

2. Second approximation

$$\sin\phi = \sin\left(\phi_s + \Delta\phi\right)$$
$$= \sin\phi_s \cos\Delta\phi + \cos\phi_s \sin\Delta\phi$$

$\Delta\phi$ small $\Rightarrow$ $\sin\phi \cong \sin\phi_s + \cos\phi_s \, \Delta\phi$

$\dfrac{d\phi_s}{dt} = 0$ by definition $\Rightarrow$ $\dfrac{d^2\phi}{dt^2} = \dfrac{d^2}{dt^2}\left(\phi_s + \Delta\phi\right) = \dfrac{d^2\Delta\phi}{dt^2}$

$\Rightarrow$

$$\frac{d^2\Delta\phi}{dt^2} + \Omega_s^{\,2} \, \Delta\phi = 0$$

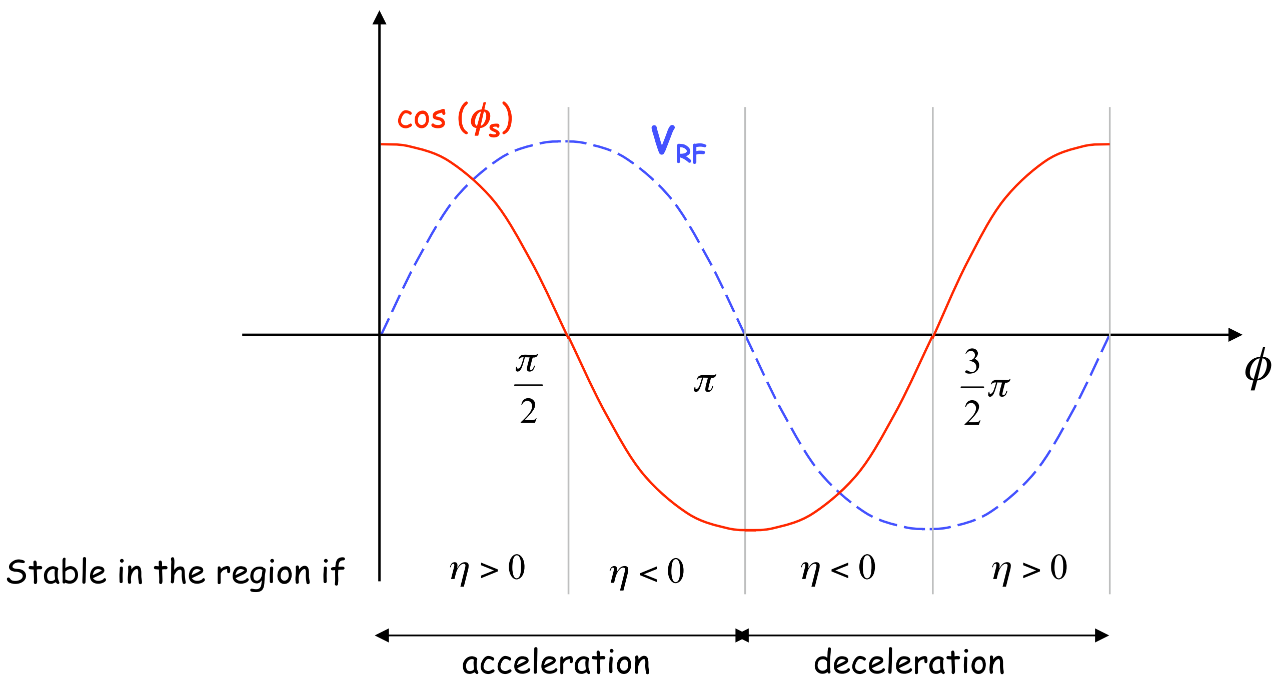
Harmonic oscillator !

SMALL AND LARGE AMPLITUDE OSCILLATIONS (2/7)

Stability condition for ϕ_s

Stability is obtained when the angular frequency of the oscillator, $\Omega_s^{\;2}$ is real positive:

$$\Omega_s^{\;2} = \frac{e\hat{V}_{RF}\,\eta\,h\,c^2}{2\pi\,R_s^{\;2}E_s}\cos\phi_s \quad \Rightarrow \quad \Omega_s^{\;2} > 0 \quad \Leftrightarrow \quad \eta\cos\phi_s > 0$$



SMALL AND LARGE AMPLITUDE OSCILLATIONS (3/7)

Small amplitude oscillations - orbits

For $\eta \cos\phi_s > 0$ the motion around the synchronous particle is a stable oscillation:

$$\begin{cases} \Delta\phi = \Delta\phi_{\max} \sin\left(\Omega_s t + \phi_0\right) \\ \Delta p = \Delta p_{\max} \cos\left(\Omega_s t + \phi_0\right) \end{cases}$$

with $$\Delta p_{\max} = \frac{\Omega_s}{B} \Delta\phi_{\max}$$

SMALL AND LARGE AMPLITUDE OSCILLATIONS (4/7)

Lepton machines

e+, e-

$\beta \cong 1$, γ large , $\eta \cong -\alpha_p$

$$\omega_s \cong \frac{c}{R_s} \quad , \quad p_s \cong \frac{E_s}{c}$$

$$\Omega_s = \frac{c}{R_s} \left\{ - \frac{e \hat{V}_{RF} \alpha_p h}{2\pi E_s} \cos\phi_s \right\}^{1/2}$$

Number of synchrotron oscillations per turn:

$$Q_s = \frac{\Omega_s}{\omega_s} = \left\{ - \frac{e \hat{V}_{RF} \alpha_p h}{2\pi E_s} \cos\phi_s \right\}^{1/2}$$

"synchrotron tune"

N.B: in these machines, the RF frequency does not change

SMALL AND LARGE AMPLITUDE OSCILLATIONS (5/7)

Large amplitude oscillations

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

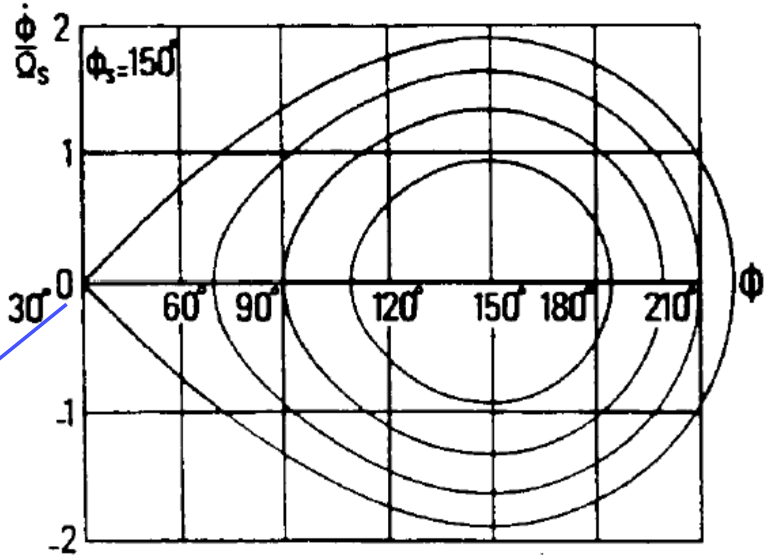
Multiplying by $\dot{\phi}$ and integrating

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = cte$$

Constant of motion

here $\dot{\phi} = 0$

$\phi = \pi - \phi_s$



Synchronous phase 150°

Equation of the separatrix

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\ \sin\phi_s\right) = -\frac{\Omega_s^2}{\cos\phi_s}\left[\cos\left(\pi - \phi_s\right) + \left(\pi - \phi_s\right)\sin\phi_s\right]$$

SMALL AND LARGE AMPLITUDE OSCILLATIONS (6/7)

"total energy"

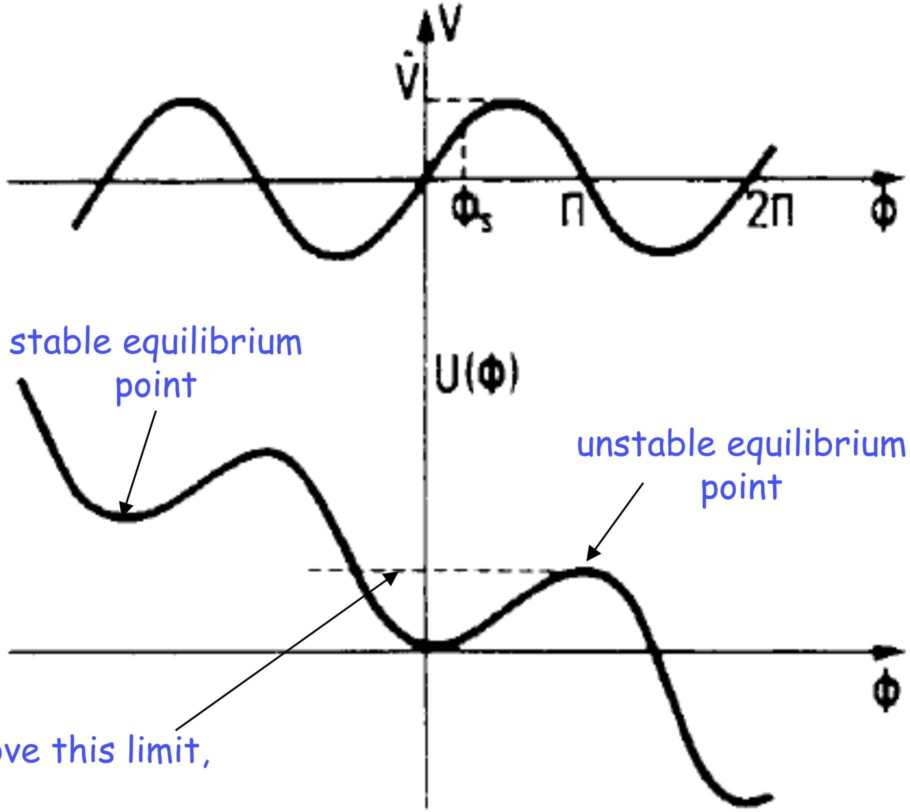
$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = cte$$

"kinetic energy" "potential energy U "

$$\frac{d^2\phi}{dt^2} = F(\phi)$$

$$F(\phi) = -\frac{\partial U}{\partial \phi}$$

Energy diagram



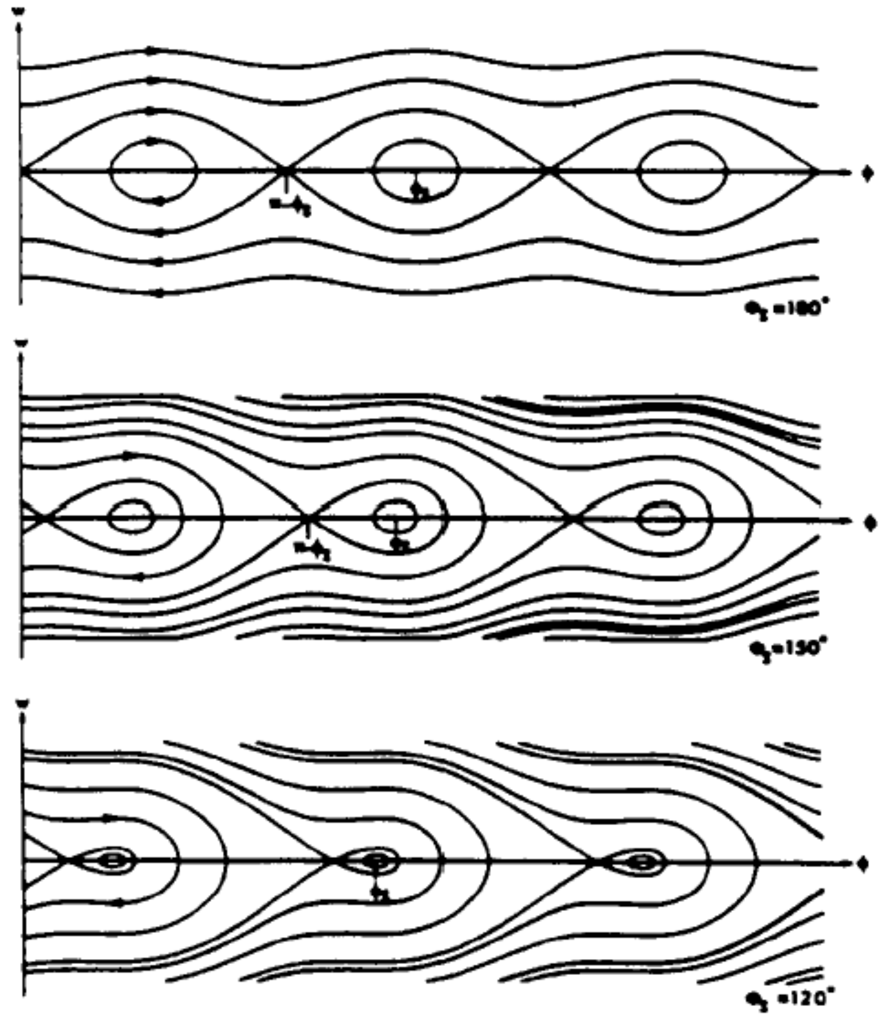
stable equilibrium point

unstable equilibrium point

If the total energy is above this limit, the motion is unbounded

SMALL AND LARGE AMPLITUDE OSCILLATIONS (7/7)

Phase space trajectories



Phase space trajectories for different synchronous phases

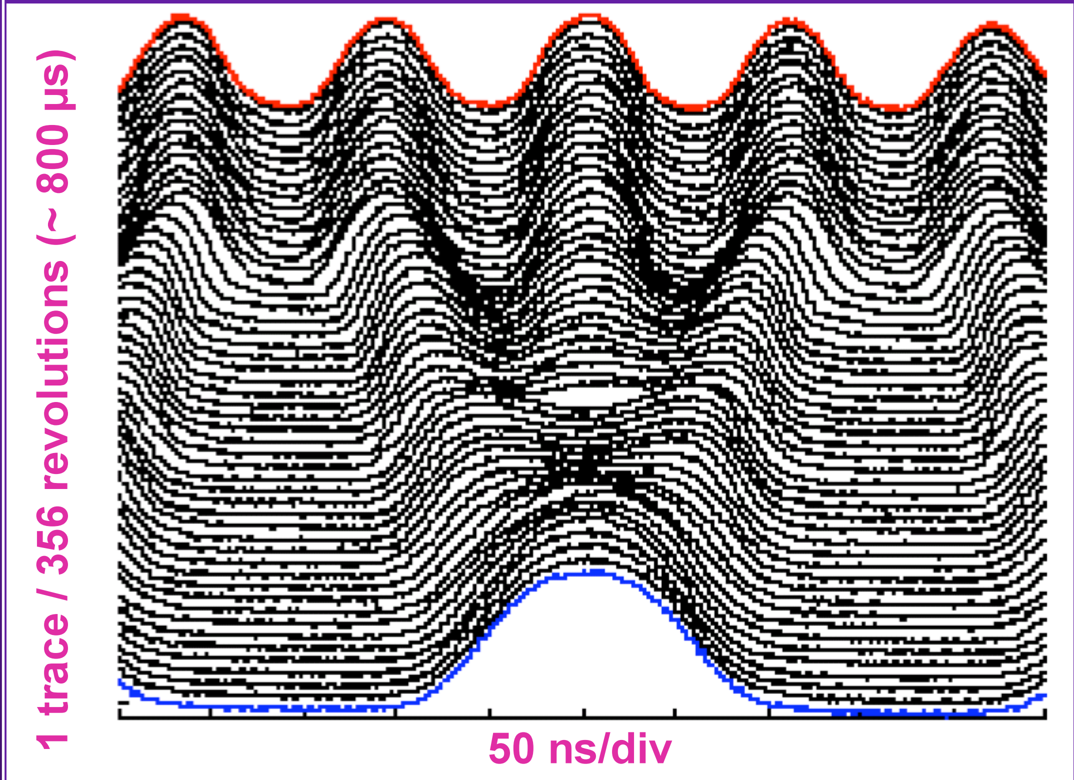
$\gamma > \gamma_{tr}$

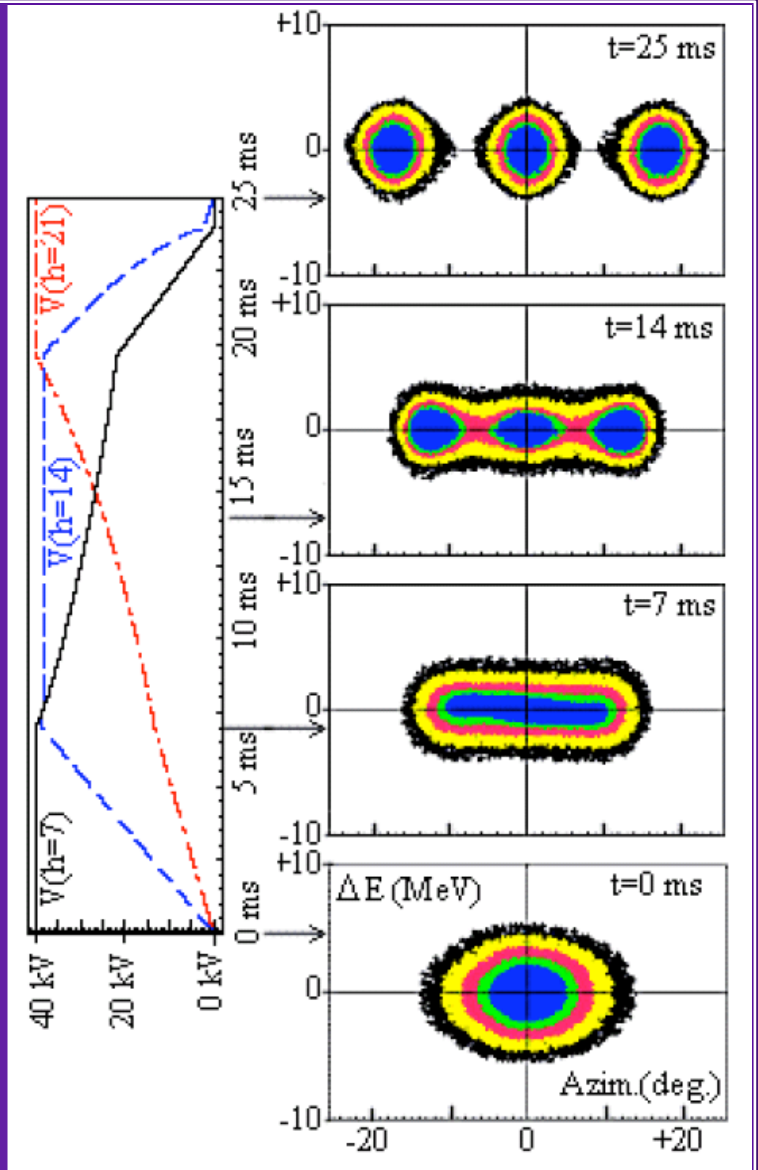
EXAMPLES OF RF MANIPULATIONS (1/2)

Double splitting also done

Triple bunch splitting

Courtesy R. Garoby





EXAMPLES OF RF MANIPULATIONS (2/2)

BUNCH ROTATION (with ESME)