# LONGITUDINAL BEAM DYNAMICS

Elias Métral

BE Department – CERN

The present transparencies are inherited from Frank Tecker (CERN-BE), who gave this course two years ago (I already gave this course in 2011&12) and who inherited them from Roberto Corsini (CERN-BE), who gave this course in the previous years, based on the ones written by Louis Rinolfi (CERN-BE) who held the course at JUAS from 1994 to 2002 (see CERN/PS 2000-008 (LP)):
http://cdsweb.cern.ch/record/446961/files/ps-2000-008.pdf

Material from Joel LeDuff's Course at the CERN Accelerator School held at Jyvaskyla, Finland the 7-18 September 1992 (CERN 94-01) has been used as well:
http://cdsweb.cern.ch/record/235242/files/p253.pdf
http://cdsweb.cern.ch/record/235242/files/p289.pdf

I attended the course given by Louis Rinolfi in 1996 and was his assistant in 2000 and 2001 (and the assistant of Michel Martini for his course on transverse beam dynamics)

This course and related exercises (as well as other courses) can be found in my web page:
http://emetral.web.cern.ch/emetral/

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## 8 Lectures

## 4 Tutorials

**Fields & Forces**
**Relativity**
**Acceleration (electrostatic, RF)**
**Synchrotons**
**Longitudinal phase space**
**Momentum Compaction**
**Transition energy**
**Synchrotron oscillations**

Examination: WE 06/02/2013
(09:00 to 10:30)

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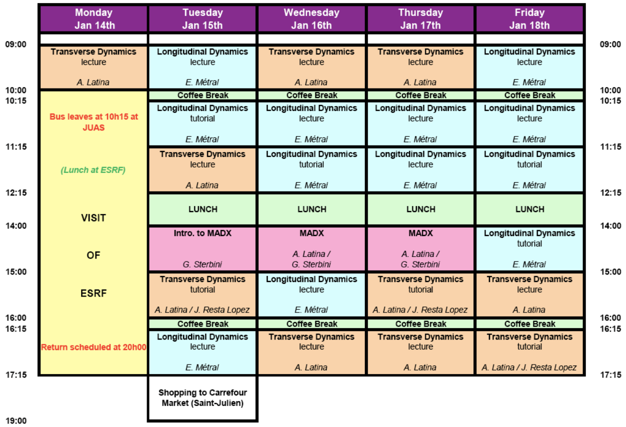
## WEEK 2



| | Monday<br>Jan 14th | Tuesday<br>Jan 15th | Wednesday<br>Jan 16th | Thursday<br>Jan 17th | Friday<br>Jan 18th | |
|---|---|---|---|---|---|---|
| 09:00 | **Transverse Dynamics**<br>lecture<br>*A. Latina* | **Longitudinal Dynamics**<br>lecture<br>*E. Métral* | **Transverse Dynamics**<br>lecture<br>*A. Latina* | **Transverse Dynamics**<br>lecture<br>*A. Latina* | **Longitudinal Dynamics**<br>lecture<br>*E. Métral* | 09:00 |
| 10:00 | **Bus leaves at 10h15 at JUAS** | **Coffee Break** | **Coffee Break** | **Coffee Break** | **Coffee Break** | 10:00 |
| 10:15 | | **Longitudinal Dynamics**<br>tutorial<br>*E. Métral* | **Longitudinal Dynamics**<br>lecture<br>*E. Métral* | **Longitudinal Dynamics**<br>lecture<br>*E. Métral* | **Longitudinal Dynamics**<br>lecture<br>*E. Métral* | 10:15 |
| 11:15 | *(Lunch at ESRF)* | **Transverse Dynamics**<br>lecture<br>*A. Latina* | **Longitudinal Dynamics**<br>tutorial<br>*E. Métral* | **Longitudinal Dynamics**<br>lecture<br>*E. Métral* | **Longitudinal Dynamics**<br>tutorial<br>*E. Métral* | 11:15 |
| 12:15 | **VISIT** | **LUNCH** | **LUNCH** | **LUNCH** | **LUNCH** | 12:15 |
| 14:00 | **OF** | **Intro. to MADX**<br>*G. Sterbini* | **MADX**<br>*A. Latina /<br>G. Sterbini* | **MADX**<br>*A. Latina /<br>G. Sterbini* | **Longitudinal Dynamics**<br>tutorial<br>*E. Métral* | 14:00 |
| 15:00 | **ESRF** | **Transverse Dynamics**<br>tutorial<br>*A. Latina / J. Resta Lopez* | **Longitudinal Dynamics**<br>lecture<br>*E. Métral* | **Transverse Dynamics**<br>tutorial<br>*A. Latina / J. Resta Lopez* | **Transverse Dynamics**<br>lecture<br>*A. Latina* | 15:00 |
| 16:00 | | **Coffee Break** | **Coffee Break** | **Coffee Break** | **Coffee Break** | 16:00 |
| 16:15 | **Return scheduled at 20h00** | **Longitudinal Dynamics**<br>lecture<br>*E. Métral* | **Transverse Dynamics**<br>lecture<br>*A. Latina* | **Transverse Dynamics**<br>lecture<br>*A. Latina* | **Transverse Dynamics**<br>tutorial<br>*A. Latina / J. Resta Lopez* | 16:15 |
| 17:15 | | **Shopping to Carrefour Market (Saint-Julien)** | | | | 17:15 |
| 19:00 | | | | | | |

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## LESSON I

Fields & forces

Acceleration by time-varying fields

Relativistic equations

## Fields and force

Equation of motion for a particle of charge **q**

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

| | |
|---|---|
| $\vec{p} = m\,\vec{v}$ | Momentum |
| $\vec{v}$ | Velocity |
| $\vec{E}$ | Electric field |
| $\vec{B}$ | Magnetic field |

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## The fields must satisfy Maxwell's equations

The integral forms, in vacuum, are recalled below:

1. Gauss's law (electrostatic) $$\int_S \vec{E}\cdot\mathrm{d}\vec{s} = \frac{1}{\varepsilon_\circ}\int_V \rho\,\mathrm{d}V$$

2. No free magnetic poles (magnetostatic) $$\int_S \vec{B}\cdot\mathrm{d}\vec{s} = 0$$

3. Ampere's law (modified by Gauss) (electric varying) $$\int_L \vec{B}\cdot\mathrm{d}\vec{l} = \mu_\circ\int_S \vec{j}\cdot\mathrm{d}\vec{s} + \frac{1}{c^2}\int_S \frac{\partial\vec{E}}{\partial t}\cdot\mathrm{d}\vec{s}$$

4. Faraday's law (magnetic varying) $$\int_L \vec{E}\cdot\mathrm{d}\vec{l} = -\int_S \frac{\partial\vec{B}}{\partial t}\cdot\mathrm{d}\vec{s}$$

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## Maxwell's equations

The differential forms, in vacuum, are recalled below:

1. Gauss's law $$\nabla\cdot\vec{E} = \frac{1}{\varepsilon_\circ}\rho(\vec{r},t)$$

2. No free magnetic poles $$\nabla\cdot\vec{B} = 0$$

3. Ampere's law (modified by Gauss) $$\nabla\times\vec{B} = \mu_\circ\,\vec{j}(\vec{r},t) + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

4. Faraday's law $$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

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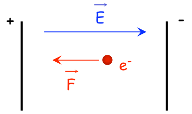
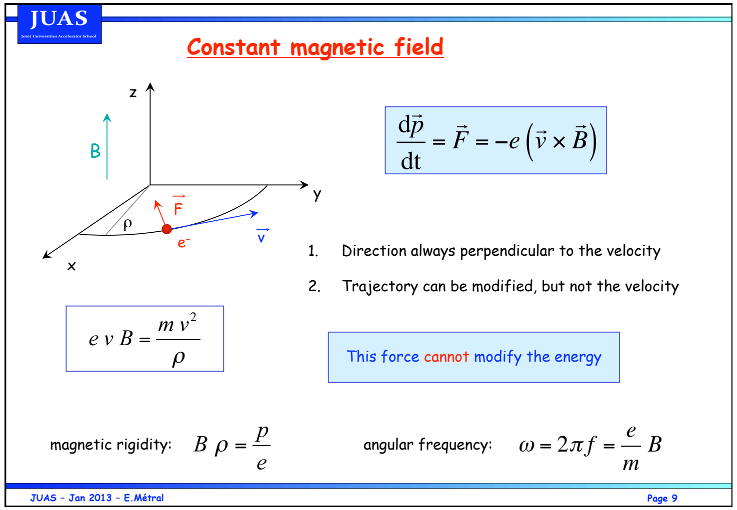
## Constant electric field



$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = -e\,\vec{E}$$

1. Direction of the force always parallel to the field
2. Trajectory can be modified, velocity also ⇒ momentum and energy can be modified

This force can be used to accelerate and decelerate particles



## Constant magnetic field

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{F} = -e\left(\vec{v}\times\vec{B}\right)$$

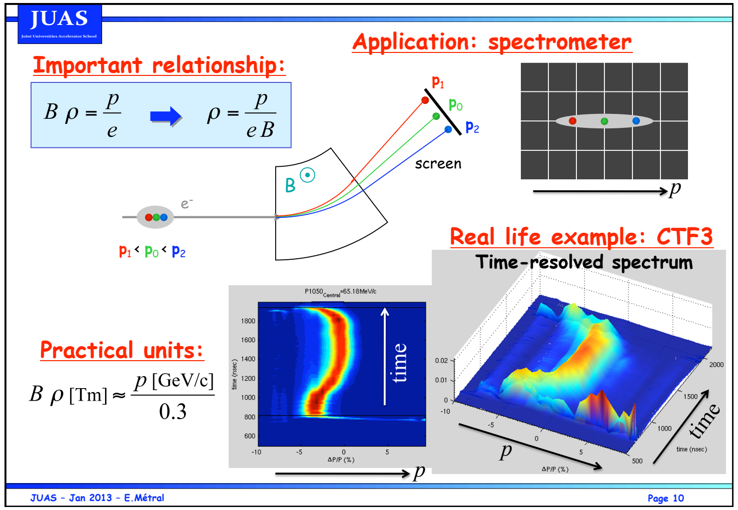
1. Direction always perpendicular to the velocity
2. Trajectory can be modified, but not the velocity

$$e\,v\,B = \frac{m\,v^2}{\rho}$$

This force cannot modify the energy

magnetic rigidity: $B\,\rho = \dfrac{p}{e}$

angular frequency: $\omega = 2\pi f = \dfrac{e}{m}B$



## Application: spectrometer

### Important relationship:

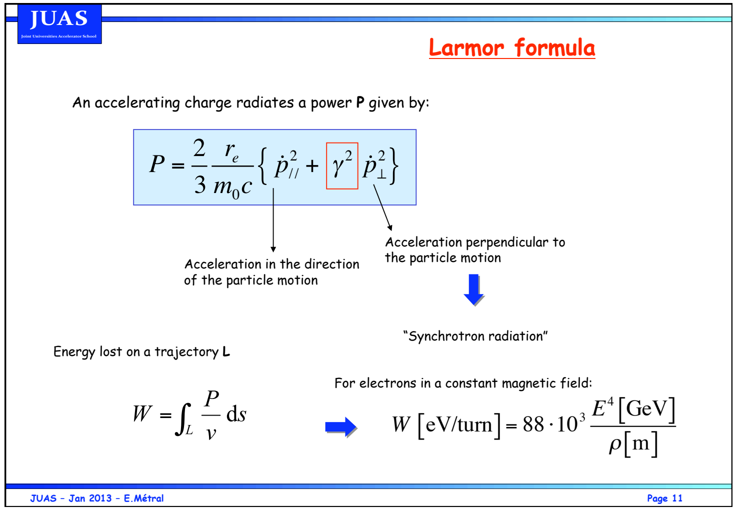
$$B\,\rho = \frac{p}{e} \Rightarrow \rho = \frac{p}{e\,B}$$

### Real life example: CTF3

Time-resolved spectrum

### Practical units:

$$B\,\rho\,[\mathrm{Tm}] \approx \frac{p\,[\mathrm{GeV/c}]}{0.3}$$



## Larmor formula

An accelerating charge radiates a power **P** given by:

$$P = \frac{2}{3}\frac{r_e}{m_0 c}\left\{\underbrace{\dot{p}_{//}^2}_{\text{Acceleration in the direction of the particle motion}} + \boxed{\gamma^2}\,\dot{p}_{\perp}^2\right\}$$

Acceleration in the direction of the particle motion

Acceleration perpendicular to the particle motion

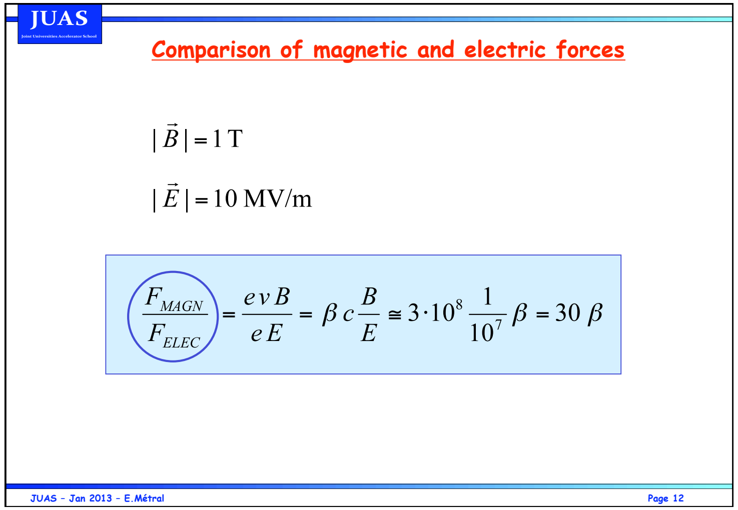
"Synchrotron radiation"

Energy lost on a trajectory **L**

$$W = \int_L \frac{P}{v}\,\mathrm{d}s$$

For electrons in a constant magnetic field:

$$\Rightarrow W\,[\mathrm{eV/turn}] = 88\cdot 10^3\,\frac{E^4\,[\mathrm{GeV}]}{\rho\,[\mathrm{m}]}$$

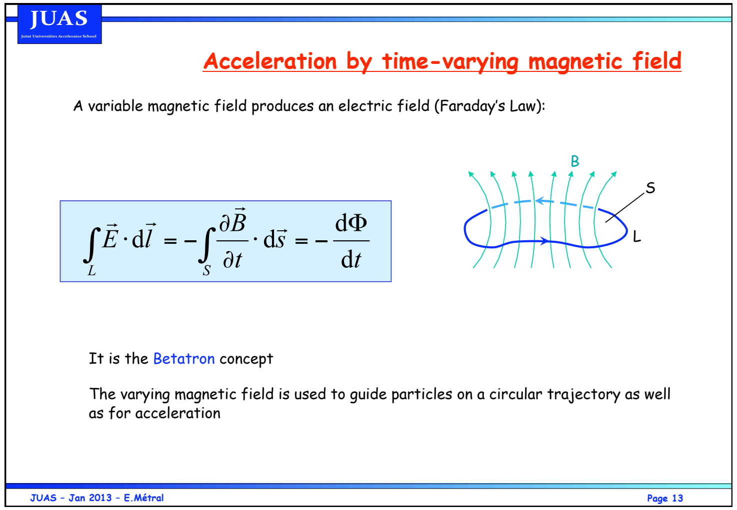


## Comparison of magnetic and electric forces

$$|\,\vec{B}\,| = 1\ \mathrm{T}$$

$$|\,\vec{E}\,| = 10\ \mathrm{MV/m}$$

$$\frac{F_{MAGN}}{F_{ELEC}} = \frac{e\,v\,B}{e\,E} = \beta\,c\,\frac{B}{E} \cong 3\cdot 10^8\,\frac{1}{10^7}\,\beta = 30\,\beta$$



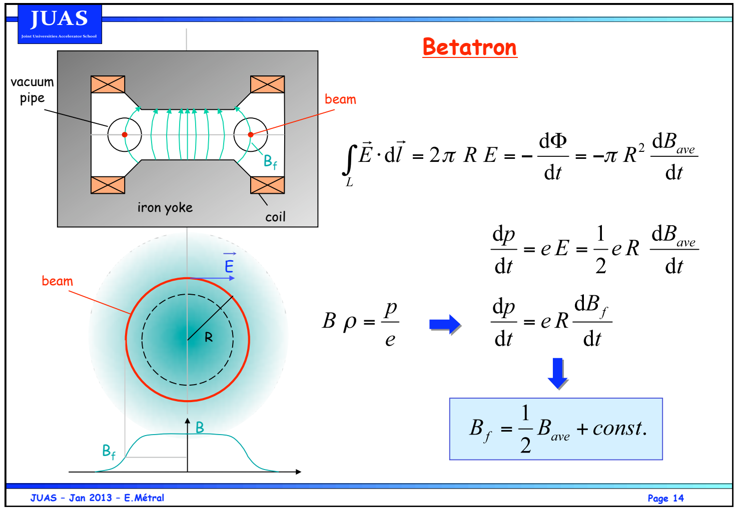
## Acceleration by time-varying magnetic field

A variable magnetic field produces an electric field (Faraday's Law):

$$\boxed{\int_L \vec{E}\cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s} = -\frac{d\Phi}{dt}}$$

It is the Betatron concept

The varying magnetic field is used to guide particles on a circular trajectory as well as for acceleration



## Betatron

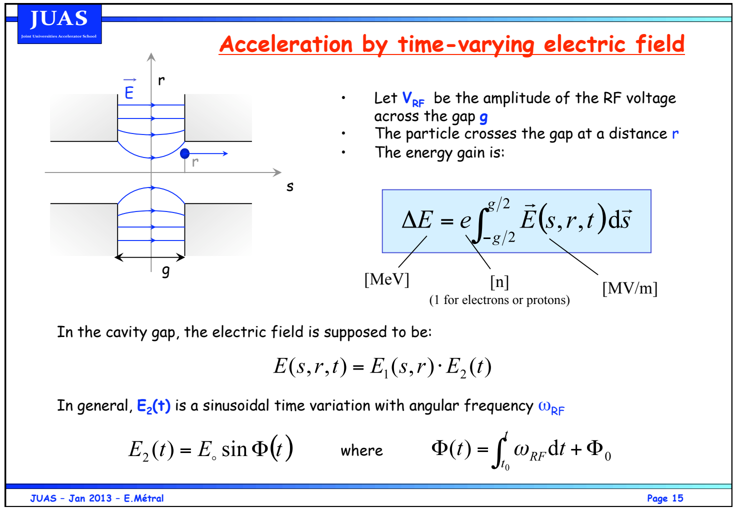
$$\int_L \vec{E}\cdot d\vec{l} = 2\pi\ R\ E = -\frac{d\Phi}{dt} = -\pi\ R^2\ \frac{dB_{ave}}{dt}$$

$$\frac{dp}{dt} = e\,E = \frac{1}{2}\,e\,R\ \frac{dB_{ave}}{dt}$$

$$B\ \rho = \frac{p}{e} \quad \Rightarrow \quad \frac{dp}{dt} = e\,R\,\frac{dB_f}{dt}$$

$$\Downarrow$$

$$\boxed{B_f = \frac{1}{2} B_{ave} + const.}$$



## Acceleration by time-varying electric field

- Let $V_{RF}$ be the amplitude of the RF voltage across the gap **g**
- The particle crosses the gap at a distance r
- The energy gain is:

$$\boxed{\Delta E = e\int_{-g/2}^{g/2} \vec{E}\left(s,r,t\right) d\vec{s}}$$

[MeV]

[n]
(1 for electrons or protons)

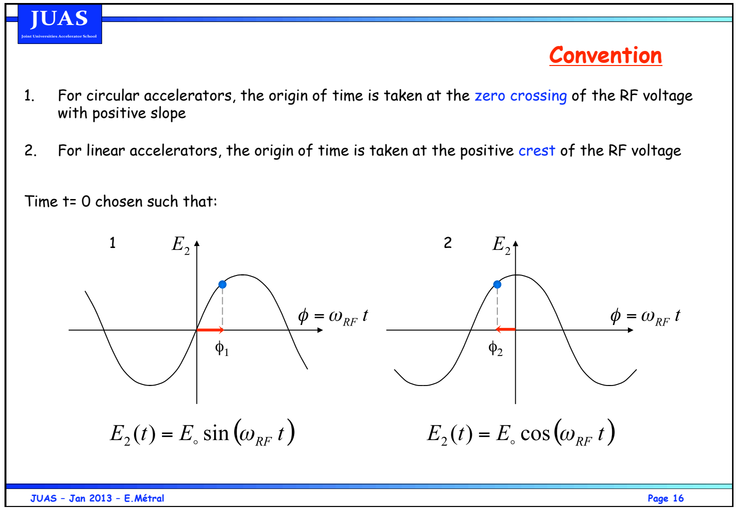
[MV/m]

In the cavity gap, the electric field is supposed to be:

$$E(s,r,t) = E_1(s,r)\cdot E_2(t)$$

In general, **$E_2(t)$** is a sinusoidal time variation with angular frequency $\omega_{RF}$

$$E_2(t) = E_\circ \sin\Phi\left(t\right) \qquad \text{where} \qquad \Phi(t) = \int_{t_0}^{t} \omega_{RF} dt + \Phi_0$$

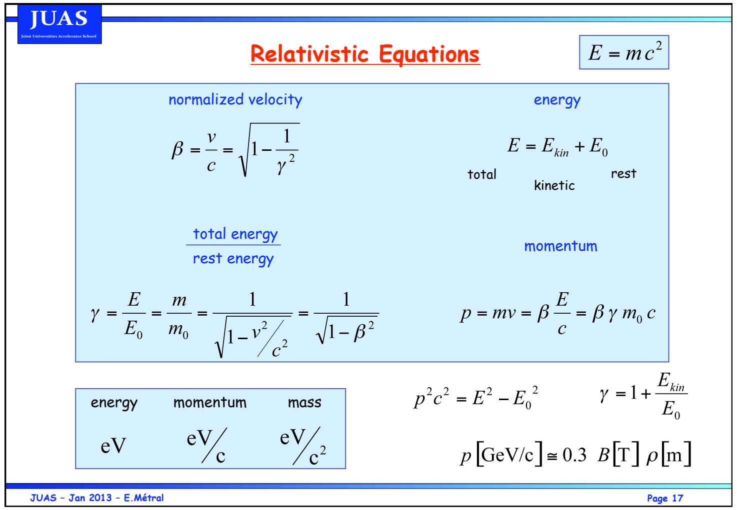


## Convention

1. For circular accelerators, the origin of time is taken at the zero crossing of the RF voltage with positive slope
2. For linear accelerators, the origin of time is taken at the positive crest of the RF voltage

Time t= 0 chosen such that:

$$E_2(t) = E_\circ \sin\left(\omega_{RF}\ t\right) \qquad E_2(t) = E_\circ \cos\left(\omega_{RF}\ t\right)$$



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## Relativistic Equations

$E = m c^2$

normalized velocity

$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}$$

energy

$$\underset{\text{total}}{E} = \underset{\text{kinetic}}{E_{kin}} + \underset{\text{rest}}{E_0}$$

$\frac{\text{total energy}}{\text{rest energy}}$

$$\gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}}$$

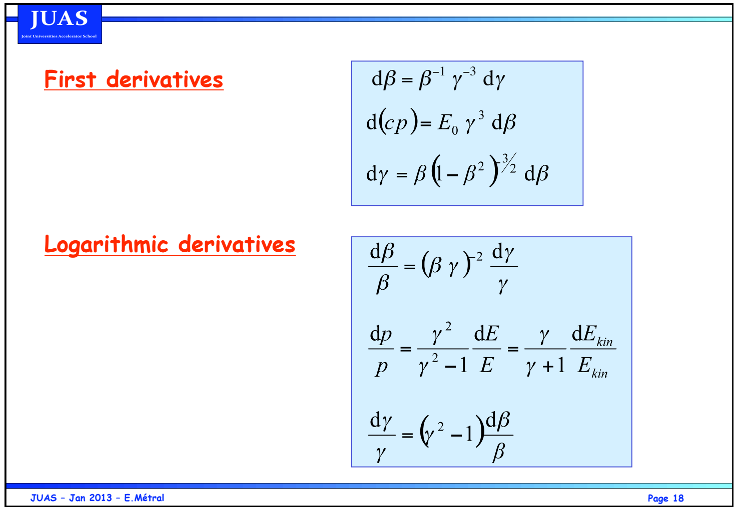
momentum

$$p = mv = \beta \frac{E}{c} = \beta \gamma m_0 c$$

| energy | momentum | mass |
|---|---|---|
| $\mathrm{eV}$ | $\mathrm{eV}/\mathrm{c}$ | $\mathrm{eV}/\mathrm{c}^2$ |

$$p^2 c^2 = E^2 - E_0^2 \qquad \gamma = 1 + \frac{E_{kin}}{E_0}$$

$$p\,[\mathrm{GeV/c}] \cong 0.3\ B\,[\mathrm{T}]\ \rho\,[\mathrm{m}]$$



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## First derivatives

$$\mathrm{d}\beta = \beta^{-1} \gamma^{-3}\, \mathrm{d}\gamma$$

$$\mathrm{d}(cp) = E_0\, \gamma^3\, \mathrm{d}\beta$$

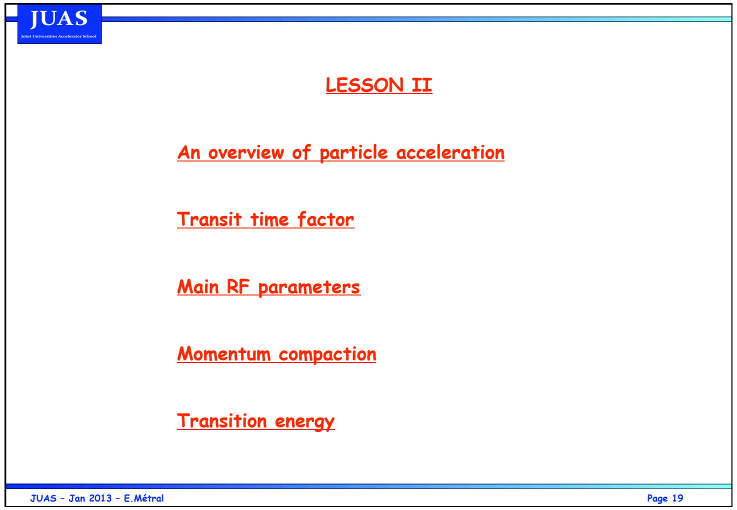
$$\mathrm{d}\gamma = \beta \left(1 - \beta^2\right)^{-3/2} \mathrm{d}\beta$$

## Logarithmic derivatives

$$\frac{\mathrm{d}\beta}{\beta} = (\beta\gamma)^{-2} \frac{\mathrm{d}\gamma}{\gamma}$$

$$\frac{\mathrm{d}p}{p} = \frac{\gamma^2}{\gamma^2 - 1} \frac{\mathrm{d}E}{E} = \frac{\gamma}{\gamma + 1} \frac{\mathrm{d}E_{kin}}{E_{kin}}$$

$$\frac{\mathrm{d}\gamma}{\gamma} = \left(\gamma^2 - 1\right) \frac{\mathrm{d}\beta}{\beta}$$



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## LESSON II

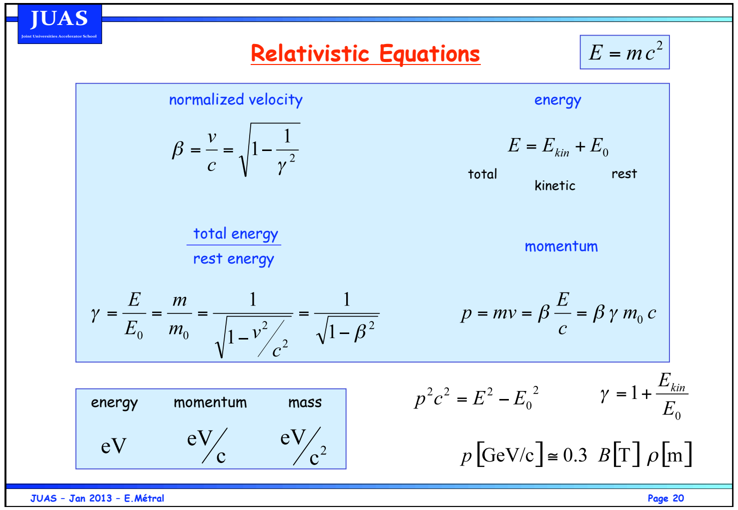
An overview of particle acceleration

Transit time factor

Main RF parameters

Momentum compaction

Transition energy



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Relativistic Equations

$E = m c^2$

normalized velocity

$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}$$

energy

$$\underset{\text{total}}{E} = \underset{\text{kinetic}}{E_{kin}} + \underset{\text{rest}}{E_0}$$

$\frac{\text{total energy}}{\text{rest energy}}$

$$\gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}}$$

momentum

$$p = mv = \beta \frac{E}{c} = \beta \gamma m_0 c$$

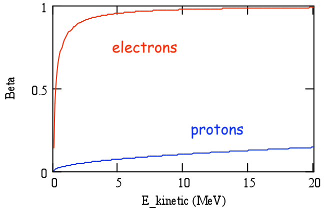
energy momentum mass
$\mathrm{eV}$ $\mathrm{eV}/\mathrm{c}$ $\mathrm{eV}/\mathrm{c}^2$

$$p^2 c^2 = E^2 - E_0^2 \qquad \gamma = 1 + \frac{E_{kin}}{E_0}$$

$$p\,[\mathrm{GeV/c}] \cong 0.3\ B\,[\mathrm{T}]\ \rho\,[\mathrm{m}]$$

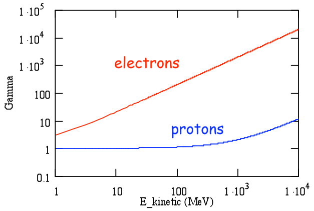
normalized velocity

$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}$$



total energy / rest energy

$$\gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}}$$



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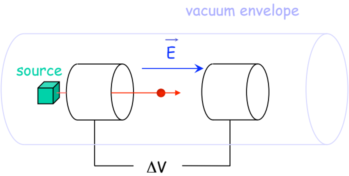
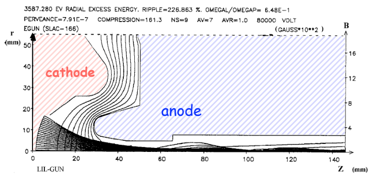
## Electrostatic accelerators



- The potential difference between two electrodes is used to accelerate particles
- Limited in energy by the maximum high voltage (~ 10 MV)
- Present applications: x-ray tubes, low energy ions, electron sources (thermionic guns)



Electric field potential and beam trajectories inside an electron gun (LEP Injector Linac at CERN), computed with the code E-GUN

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Electrostatic accelerator

Protons & Ions

750 kV Cockroft-Walton source of LINAC 2 (CERN) © CERN Geneva

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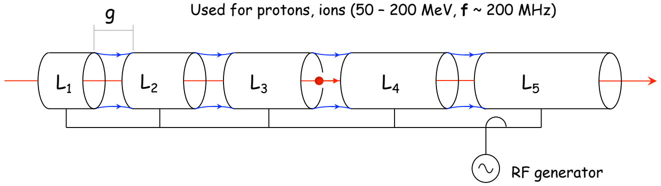
## Alvarez structure

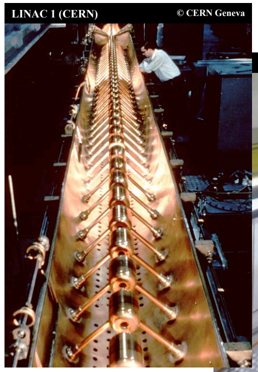
Used for protons, ions (50 - 200 MeV, **f** ~ 200 MHz)

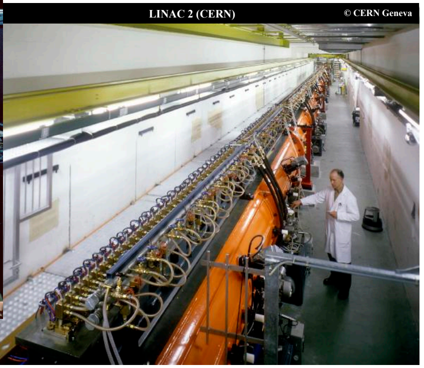


Synchronism condition $(g << L)$

$$\Rightarrow \boxed{L = v_s\, T_{RF} = \beta_s\, \lambda_{RF}} \qquad \omega_{RF} = 2\pi \frac{v_s}{L}$$

## Proton and ion linacs
(Alvarez structure)


LINAC 1 (CERN) © CERN Geneva


LINAC 2 (CERN) © CERN Geneva

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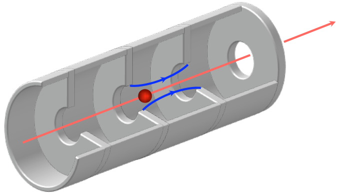
## Electron Linac



Electrons are light ⇒ fast acceleration
⇒ β ≅ 1 already at an energy of a few MeV
⬇
Uniform disk-loaded waveguide, travelling wave
(up to 50 GeV, **f** ~ 3 GHz - S-band)

$$E(z,t) = E_0\ e^{i(\omega t - k z)}$$ Electric field

Wave number $k = \frac{2\pi}{\lambda_{RF}}$ Phase velocity $v_{ph} = \frac{\omega}{k}$ Group velocity $v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}$

Synchronism condition ➡ $\boxed{v_{el} \approx c = \frac{\omega}{k} = v_{ph}}$

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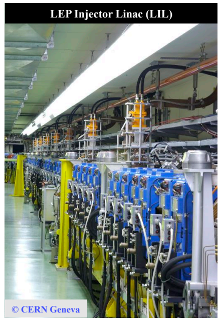
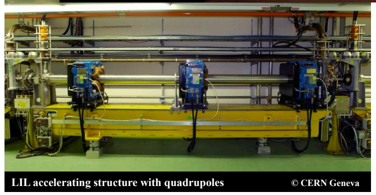
## Electron linacs & structures


LEP Injector Linac (LIL)
© CERN Geneva


CLIC Accelerating Structures (30 GHz & 11 GHz)
© CERN Geneva


LIL accelerating structure with quadrupoles © CERN Geneva

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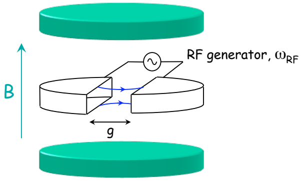
## Cyclotron

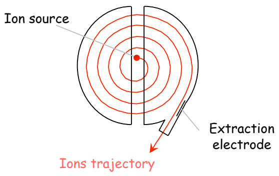
Used for protons, ions

B = constant
$\omega_{RF}$ = constant



Synchronism condition ➡

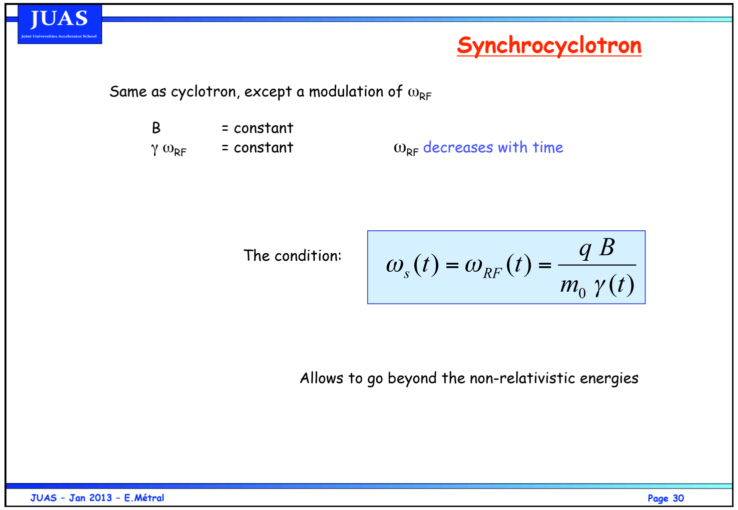
$$\boxed{\begin{aligned} \omega_s &= \omega_{RF} \\ 2\pi\ \rho &= v_s\ T_{RF} \end{aligned}}$$



Cyclotron frequency $\omega = \frac{q\ B}{m_0\ \gamma}$

1. $\gamma$ increases with the energy
   ⇒ no exact synchronism
2. if **v << c** ⇒ **γ ≅ 1**

Cyclotron ($H^-$ accelerated, protons extracted)



## Synchrocyclotron

Same as cyclotron, except a modulation of $\omega_{RF}$

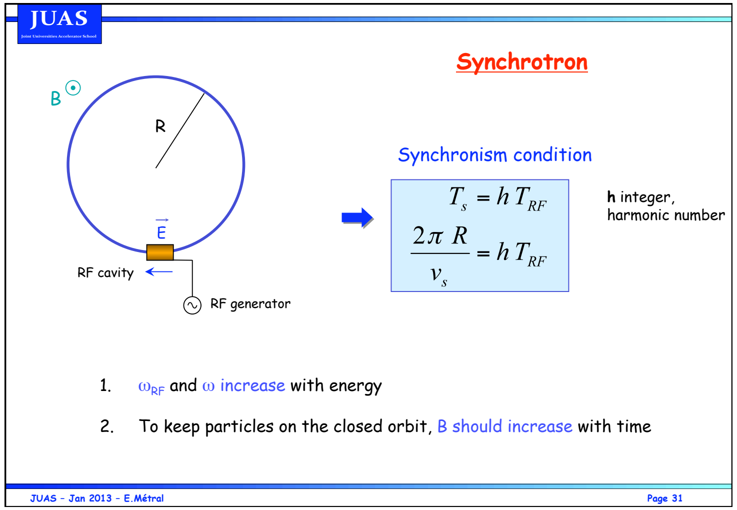
B = constant
$\gamma \, \omega_{RF}$ = constant

$\omega_{RF}$ decreases with time

The condition:

$$\omega_s(t) = \omega_{RF}(t) = \frac{q\,B}{m_0\,\gamma(t)}$$

Allows to go beyond the non-relativistic energies



## Synchrotron

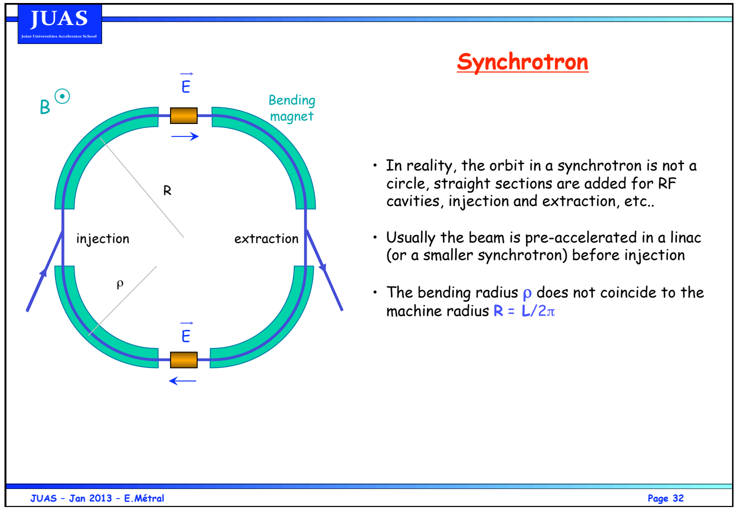
Synchronism condition

$$T_s = h\,T_{RF}$$

$$\frac{2\pi\,R}{v_s} = h\,T_{RF}$$

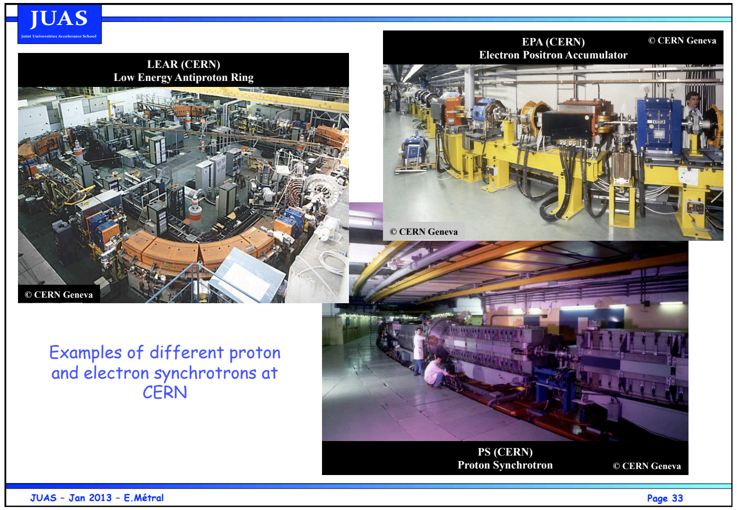
**h** integer,
harmonic number

1. $\omega_{RF}$ and $\omega$ increase with energy
2. To keep particles on the closed orbit, B should increase with time



## Synchrotron

- In reality, the orbit in a synchrotron is not a circle, straight sections are added for RF cavities, injection and extraction, etc..
- Usually the beam is pre-accelerated in a linac (or a smaller synchrotron) before injection
- The bending radius $\rho$ does not coincide to the machine radius $R = L/2\pi$



LEAR (CERN)
Low Energy Antiproton Ring
© CERN Geneva

EPA (CERN)
Electron Positron Accumulator
© CERN Geneva

PS (CERN)
Proton Synchrotron
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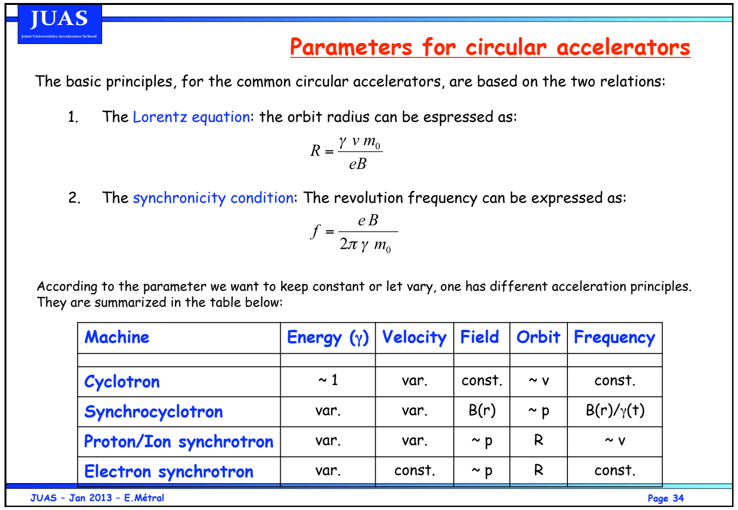
Examples of different proton and electron synchrotrons at CERN



## Parameters for circular accelerators

The basic principles, for the common circular accelerators, are based on the two relations:

1. The Lorentz equation: the orbit radius can be espressed as:

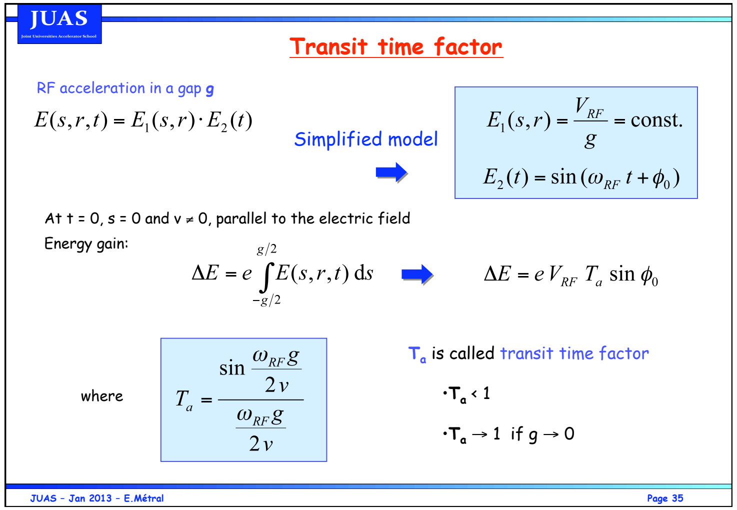
$$R = \frac{\gamma\, v\, m_0}{eB}$$

2. The synchronicity condition: The revolution frequency can be expressed as:

$$f = \frac{e\,B}{2\pi\, \gamma\, m_0}$$

According to the parameter we want to keep constant or let vary, one has different acceleration principles. They are summarized in the table below:

| Machine | Energy (γ) | Velocity | Field | Orbit | Frequency |
|---|---|---|---|---|---|
| **Cyclotron** | ~ 1 | var. | const. | ~ v | const. |
| **Synchrocyclotron** | var. | var. | B(r) | ~ p | B(r)/γ(t) |
| **Proton/Ion synchrotron** | var. | var. | ~ p | R | ~ v |
| **Electron synchrotron** | var. | const. | ~ p | R | const. |



## Transit time factor

RF acceleration in a gap **g**

$$E(s,r,t) = E_1(s,r) \cdot E_2(t)$$

Simplified model ➡

$$E_1(s,r) = \frac{V_{RF}}{g} = \text{const.}$$

$$E_2(t) = \sin(\omega_{RF}\, t + \phi_0)$$

At t = 0, s = 0 and v ≠ 0, parallel to the electric field

Energy gain:

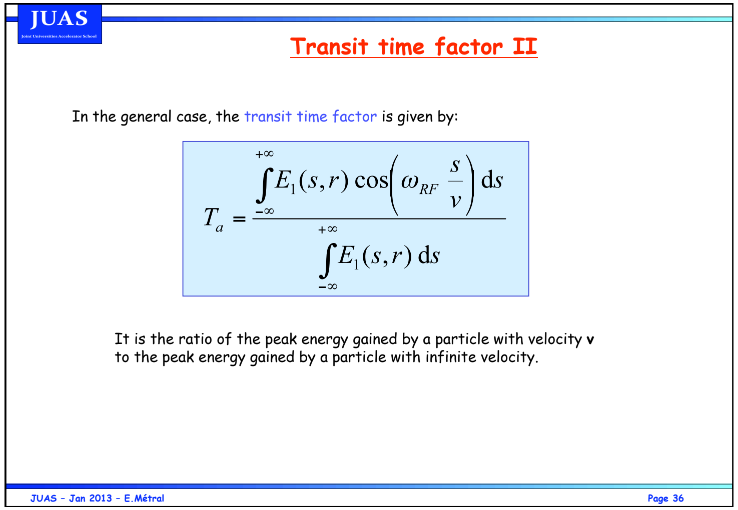
$$\Delta E = e \int_{-g/2}^{g/2} E(s,r,t)\, \mathrm{d}s \quad ➡ \quad \Delta E = e\, V_{RF}\, T_a \sin \phi_0$$

where

$$T_a = \frac{\sin \dfrac{\omega_{RF}\, g}{2\,v}}{\dfrac{\omega_{RF}\, g}{2\,v}}$$

$T_a$ is called transit time factor

- $T_a < 1$
- $T_a \rightarrow 1$ if $g \rightarrow 0$



## Transit time factor II

In the general case, the transit time factor is given by:

$$T_a = \frac{\int_{-\infty}^{+\infty} E_1(s,r) \cos\left(\omega_{RF} \frac{s}{v}\right) \mathrm{d}s}{\int_{-\infty}^{+\infty} E_1(s,r)\, \mathrm{d}s}$$

It is the ratio of the peak energy gained by a particle with velocity **v** to the peak energy gained by a particle with infinite velocity.

## Main RF parameters

### I. Voltage, phase, frequency

In order to accelerate particles, longitudinal fields must be generated in the direction of the desired acceleration

$$E(s,t) = E_1(s) \cdot E_2(t) \qquad E_2(t) = E_0 \sin\left[\int_{t_0}^{t} \omega_{RF}\, \mathrm{d}t + \phi_0\right]$$

$$\omega_{RF} = 2\pi\, f_{RF} \qquad \Delta E = e\, V_{RF}\, T_a \sin \phi_0$$

Such electric fields are generated in RF cavities characterized by the voltage amplitude, the frequency and the phase

### II. Harmonic number

$$T_{rev} = h\, T_{RF} \quad \Rightarrow \quad f_{RF} = h\, f_{rev}$$

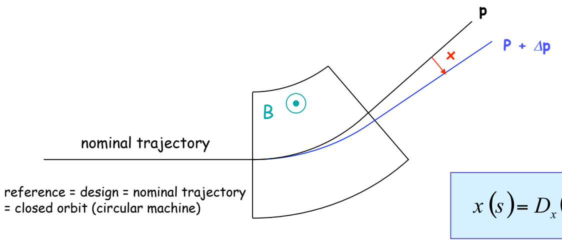
**$f_{rev}$** = revolution frequency
**$f_{RF}$** = frequency of the RF
**h** = harmonic number

harmonic number in different machines:

| AA | EPA | PS | SPS |
|---|---|---|---|
| 1 | 8 | 20 | 4620 |

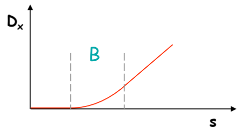
---

## Dispersion



reference = design = nominal trajectory
= closed orbit (circular machine)

$$x(s) = D_x(s)\, \frac{\Delta p}{p}$$



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## Momentum compaction factor in a transport system

In a particle transport system, a nominal trajectory is defined for the nominal momentum p.

For a particle with a momentum p + Δp the trajectory length can be different from the length L of the nominal trajectory.

The momentum compaction factor is defined by the ratio:

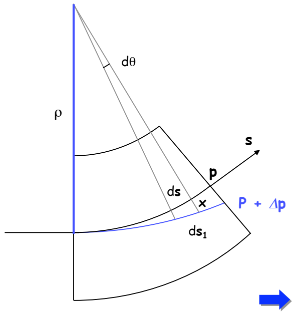
$$\alpha_p = \frac{dL/L}{dp/p}$$

Therefore, for small momentum deviation, to first order it is:

$$\frac{\Delta L}{L} = \alpha_p\, \frac{\Delta p}{p}$$

---

## Example: constant magnetic field



$$ds = \rho\, \mathrm{d}\theta$$

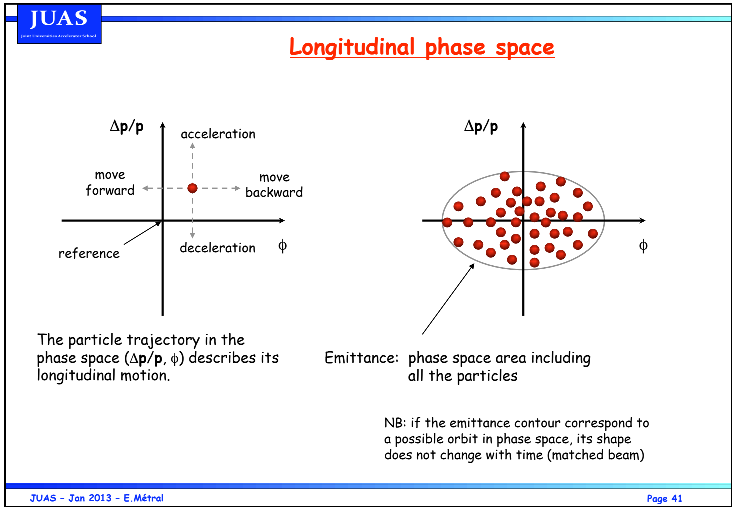
$$ds_1 = (\rho + x)\mathrm{d}\theta$$

$$\frac{ds_1 - ds}{ds} = \frac{(\rho + x)\mathrm{d}\theta - \rho\, \mathrm{d}\theta}{\rho\, \mathrm{d}\theta} = \frac{x}{\rho} \underset{\uparrow}{=} \frac{D_x}{\rho}\frac{\mathrm{d}p}{p}$$

By definition of dispersion **$D_x$**

$$\Rightarrow \quad \alpha_p = \frac{1}{L}\int_0^L \frac{D_x(s)}{\rho(s)}\, \mathrm{d}s$$

To first order, only the bending magnets contribute to a change of the trajectory length
(r = ∞ in the straight sections)

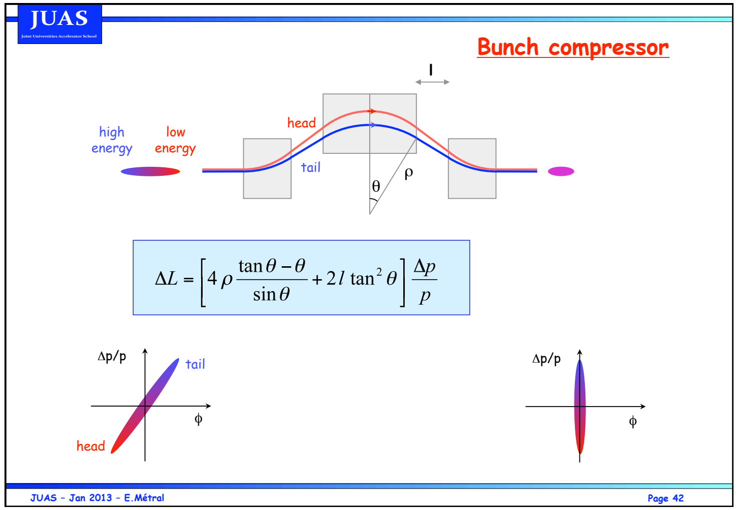


## Longitudinal phase space

The particle trajectory in the phase space (**Δp/p**, ϕ) describes its longitudinal motion.

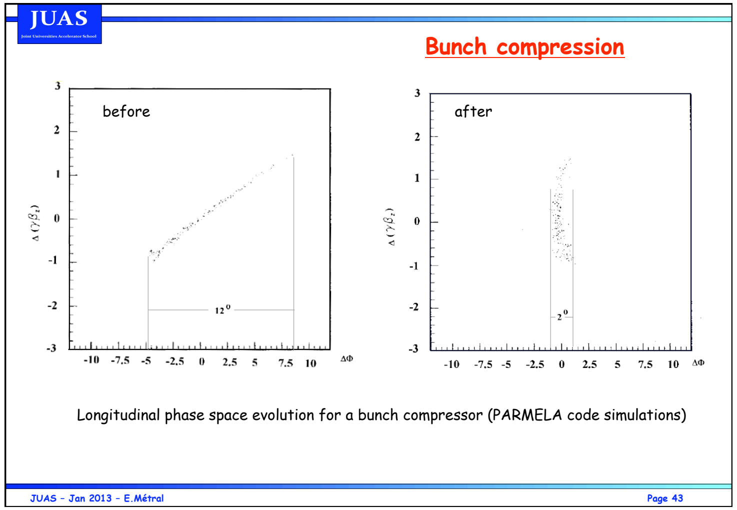
Emittance: phase space area including all the particles

NB: if the emittance contour correspond to a possible orbit in phase space, its shape does not change with time (matched beam)



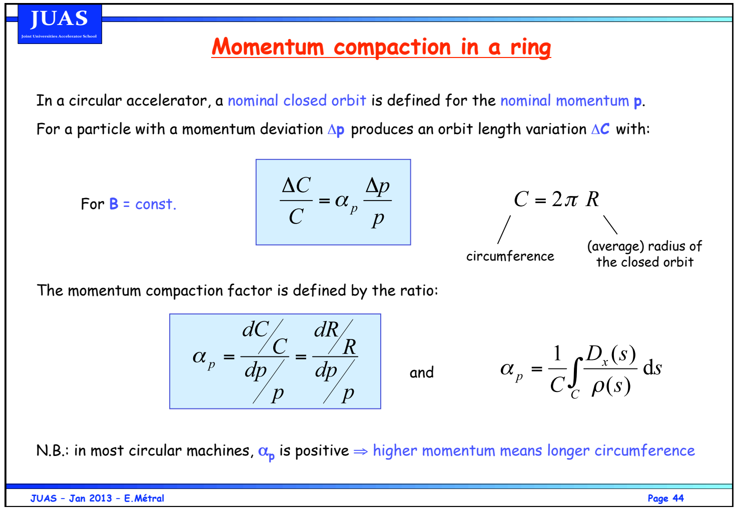
## Bunch compressor

$$\Delta L = \left[ 4\rho \frac{\tan\theta - \theta}{\sin\theta} + 2l\tan^2\theta \right] \frac{\Delta p}{p}$$



## Bunch compression

Longitudinal phase space evolution for a bunch compressor (PARMELA code simulations)



## Momentum compaction in a ring

In a circular accelerator, a nominal closed orbit is defined for the nominal momentum **p**.

For a particle with a momentum deviation **Δp** produces an orbit length variation **ΔC** with:

For **B** = const.

$$\frac{\Delta C}{C} = \alpha_p \frac{\Delta p}{p}$$

$$C = 2\pi\ R$$

circumference

(average) radius of the closed orbit

The momentum compaction factor is defined by the ratio:

$$\alpha_p = \frac{dC/C}{dp/p} = \frac{dR/R}{dp/p}$$

and

$$\alpha_p = \frac{1}{C}\int_C \frac{D_x(s)}{\rho(s)}\,\mathrm{ds}$$

N.B.: in most circular machines, $\alpha_p$ is positive ⇒ higher momentum means longer circumference

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## Momentum compaction as a function of energy

$$E = \frac{p\,c}{\beta} \;\Rightarrow\; \frac{\mathrm{d}E}{E} = \beta^2 \frac{dp}{p}$$

$$\boxed{\alpha_p = \beta^2 \frac{E}{R}\frac{\mathrm{d}R}{\mathrm{d}E}}$$

---

JUAS

## Momentum compaction as a function of magnetic field

Definition of average magnetic field

$$< B > = \frac{1}{2\pi\, R}\oint_C B_f\ \mathrm{d}s = \frac{1}{2\pi\, R}\left( \underbrace{\int_{straights} B_f\ \mathrm{d}s}_{=0} + \underbrace{\int_{magnets} B_f\ \mathrm{d}s}_{2\pi\,\rho\, B_f} \right)$$

$$< B > = \frac{B_f\,\rho}{R} \;\Rightarrow\; \frac{\mathrm{d} < B >}{< B >} = \frac{\mathrm{d}B_f}{B_f} + \frac{\mathrm{d}\rho}{\rho} - \frac{\mathrm{d}R}{R}$$

$$B_f\ \rho = \frac{p}{e} \;\Downarrow\; < B > R = \frac{p}{e} \;\Rightarrow\; \frac{\mathrm{d} < B >}{< B >} + \frac{\mathrm{d}R}{R} = \frac{\mathrm{d}\,p}{p}$$

For $B_f$ = const.

$$\boxed{\alpha_p = 1 - \frac{\mathrm{d} < B >}{< B >} \Big/ \frac{\mathrm{d}\,p}{p}}$$

---

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## Transition energy

Proton (ion) circular machine with $\alpha_p$ positive

1. Momentum larger than the nominal ( $p + \Delta p$ ) ⇒ longer orbit ( $C+\Delta C$ )
2. Momentum larger than the nominal ( $p + \Delta p$ ) ⇒ higher velocity ( $v + \Delta v$ )

What happens to the revolution frequency $f = v/C$ ?

- At low energy, $v$ increases faster than $C$ with momentum
- At high energy $v \cong c$ and remains almost constant

⇒ There is an energy for which the velocity variation is compensated by the trajectory variation ⇒ transition energy

| | | |
|---|---|---|
| Below transition: | higher energy ⇒ | higher revolution frequency |
| Above transition: | higher energy ⇒ | lower revolution frequency |

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## Transition energy - quantitative approach

We define a parameter $\eta$ (revolution frequency spread per unit of momentum spread):

$$\boxed{\eta = \frac{\mathrm{d}\,f \big/ f}{\mathrm{d}\,p \big/ p} = \frac{\mathrm{d}\omega \big/ \omega}{\mathrm{d}\,p \big/ p}}$$

$$f = \frac{v}{C} \;\Rightarrow\; \frac{\mathrm{d}\,f}{f} = \frac{\mathrm{d}\,\beta}{\beta} - \frac{\mathrm{d}\,C}{C}$$

from $p = \frac{m_0\,c\,\beta}{\sqrt{1-\beta^2}} \Rightarrow \frac{\mathrm{d}\,\beta}{\beta} = \frac{1}{\gamma^2}\frac{\mathrm{d}\,p}{p}$

definition of momentum compaction factor: $\frac{\mathrm{d}\,C}{C} = \alpha_p \frac{\mathrm{d}\,p}{p}$

$$\boxed{\frac{\mathrm{d}\,f}{f} = \left(\frac{1}{\gamma^2} - \alpha_p\right)\frac{\mathrm{d}\,p}{p}}$$

## Transition energy – quantitative approach

$$\eta = \frac{1}{\gamma^2} - \alpha_p$$

The transition energy is the energy that corresponds to $\eta = 0$ ( $\alpha_p$ is fixed, and $\gamma$ variable )

⬇

$$\gamma_{tr} = \sqrt{\frac{1}{\alpha_p}}$$

The parameter $\eta$ can also be written as

$$\eta = \frac{1}{\gamma^2} - \frac{1}{\gamma_{tr}^2}$$

- At low energy $\eta > 0$
- At high energy $\eta < 0$

N.B.: for electrons, $\gamma >> \gamma_{tr} \Rightarrow \eta < 0$
for linacs $\alpha_p = 0 \Rightarrow \eta > 0$

---

## LESSON III

Equations related to synchrotrons

Synchronous particle

Synchrotron oscillations

Principle of phase stability

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## Equations related to synchrotrons

$$\frac{\mathrm{d}p}{p} = \gamma_{tr}^2 \frac{\mathrm{d}R}{R} + \frac{\mathrm{d}B}{B}$$

$$\frac{\mathrm{d}p}{p} = \gamma^2 \frac{\mathrm{d}f}{f} + \gamma^2 \frac{\mathrm{d}R}{R}$$

$$\frac{\mathrm{d}B}{B} = \gamma_{tr}^2 \frac{\mathrm{d}f}{f} + \left[1 - \left(\frac{\gamma_{tr}}{\gamma}\right)^2\right] \frac{\mathrm{d}p}{p}$$

$$\frac{\mathrm{d}B}{B} = \gamma^2 \frac{\mathrm{d}f}{f} + \left(\gamma^2 - \gamma_{tr}^2\right) \frac{\mathrm{d}R}{R}$$

| | |
|---|---|
| $p\,[\mathrm{MeV/c}]$ | momentum |
| $R\,[\mathrm{m}]$ | orbit radius |
| $B\,[\mathrm{T}]$ | magnetic field |
| $f\,[\mathrm{Hz}]$ | rev. frequency |
| $\gamma_{tr}$ | transition energy |

---

## I - Constant radius

$$\mathrm{d}R = 0$$

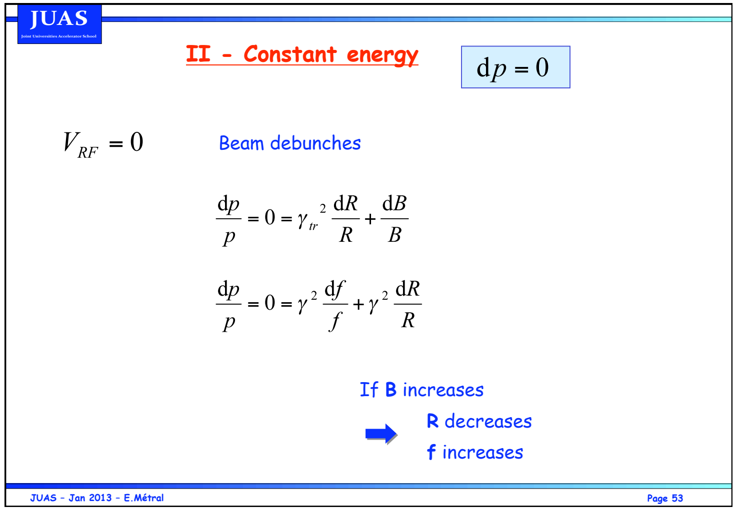
Beam maintained on the same orbit when energy varies

$$\frac{\mathrm{d}p}{p} = \frac{\mathrm{d}B}{B}$$

$$\frac{\mathrm{d}p}{p} = \gamma^2 \frac{\mathrm{d}f}{f}$$

If **p** increases

➡ **B** increases
**f** increases



## II - Constant energy

$$\mathrm{d}p = 0$$

$V_{RF} = 0$ Beam debunches

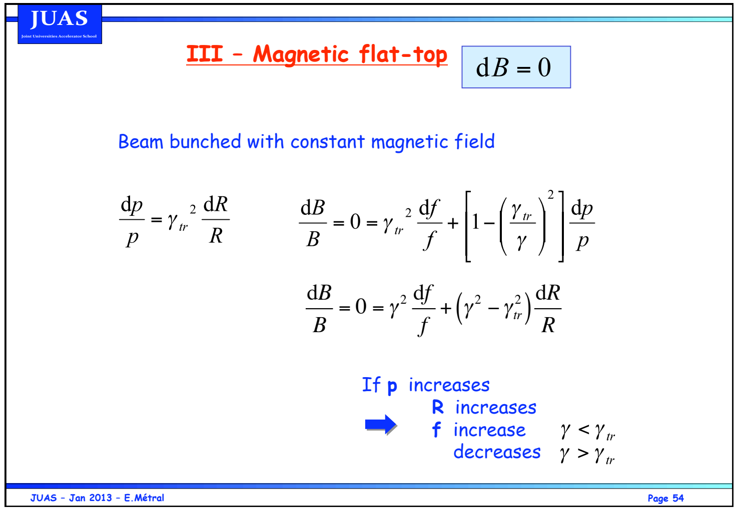
$$\frac{\mathrm{d}p}{p} = 0 = \gamma_{tr}^{\ 2}\frac{\mathrm{d}R}{R} + \frac{\mathrm{d}B}{B}$$

$$\frac{\mathrm{d}p}{p} = 0 = \gamma^2\frac{\mathrm{d}f}{f} + \gamma^2\frac{\mathrm{d}R}{R}$$

If **B** increases

➡ **R** decreases

**f** increases



## III - Magnetic flat-top

$$\mathrm{d}B = 0$$

Beam bunched with constant magnetic field

$$\frac{\mathrm{d}p}{p} = \gamma_{tr}^{\ 2}\frac{\mathrm{d}R}{R} \qquad \frac{\mathrm{d}B}{B} = 0 = \gamma_{tr}^{\ 2}\frac{\mathrm{d}f}{f} + \left[1 - \left(\frac{\gamma_{tr}}{\gamma}\right)^2\right]\frac{\mathrm{d}p}{p}$$

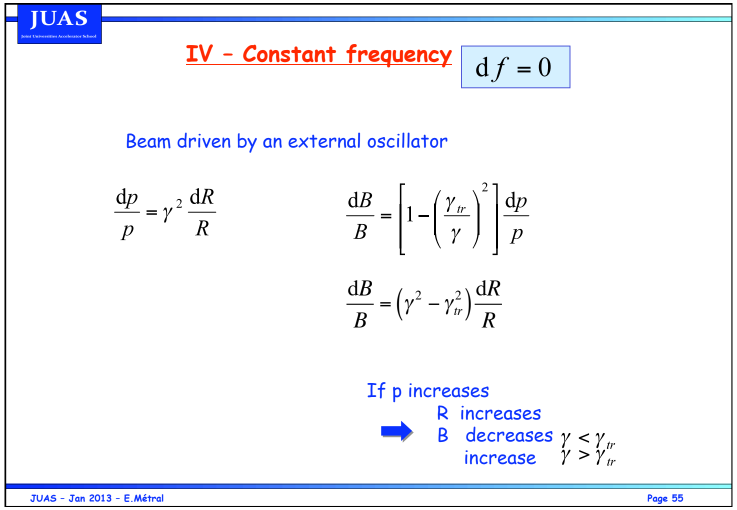
$$\frac{\mathrm{d}B}{B} = 0 = \gamma^2\frac{\mathrm{d}f}{f} + \left(\gamma^2 - \gamma_{tr}^2\right)\frac{\mathrm{d}R}{R}$$

If **p** increases

**R** increases

➡ **f** increase $\gamma < \gamma_{tr}$

decreases $\gamma > \gamma_{tr}$



## IV - Constant frequency

$$\mathrm{d}f = 0$$

Beam driven by an external oscillator

$$\frac{\mathrm{d}p}{p} = \gamma^2\frac{\mathrm{d}R}{R} \qquad \frac{\mathrm{d}B}{B} = \left[1 - \left(\frac{\gamma_{tr}}{\gamma}\right)^2\right]\frac{\mathrm{d}p}{p}$$

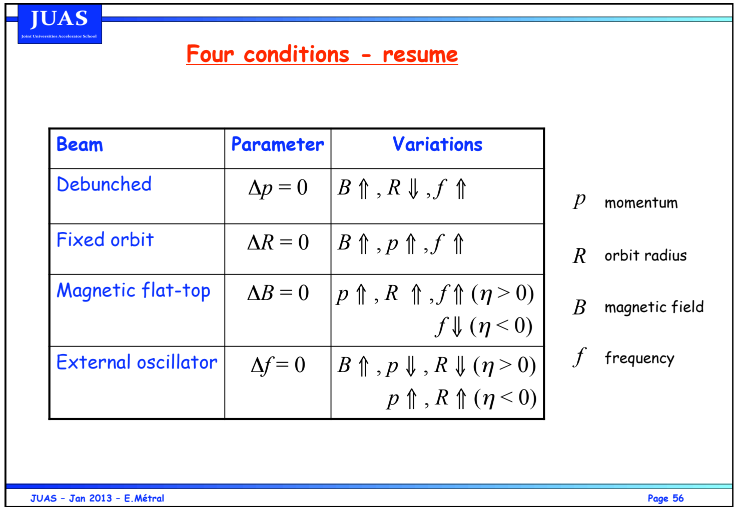
$$\frac{\mathrm{d}B}{B} = \left(\gamma^2 - \gamma_{tr}^2\right)\frac{\mathrm{d}R}{R}$$

If p increases

R increases

➡ B decreases $\gamma < \gamma_{tr}$

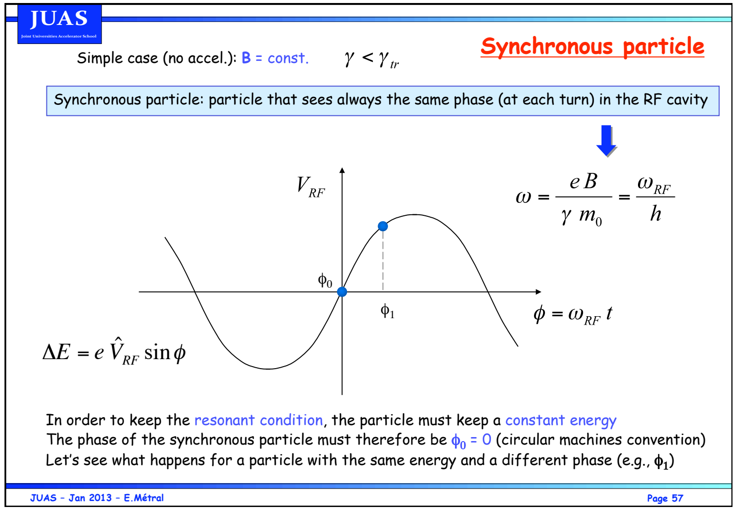
increase $\gamma > \gamma_{tr}$



## Four conditions - resume

| **Beam** | **Parameter** | **Variations** |
|---|---|---|
| Debunched | $\Delta p = 0$ | $B \Uparrow , R \Downarrow , f \Uparrow$ |
| Fixed orbit | $\Delta R = 0$ | $B \Uparrow , p \Uparrow , f \Uparrow$ |
| Magnetic flat-top | $\Delta B = 0$ | $p \Uparrow , R \Uparrow , f \Uparrow (\eta > 0)$<br>$f \Downarrow (\eta < 0)$ |
| External oscillator | $\Delta f = 0$ | $B \Uparrow , p \Downarrow , R \Downarrow (\eta > 0)$<br>$p \Uparrow , R \Uparrow (\eta < 0)$ |

| | |
|---|---|
| $p$ | momentum |
| $R$ | orbit radius |
| $B$ | magnetic field |
| $f$ | frequency |



## Synchronous particle

Simple case (no accel.): B = const. $\gamma < \gamma_{tr}$

Synchronous particle: particle that sees always the same phase (at each turn) in the RF cavity

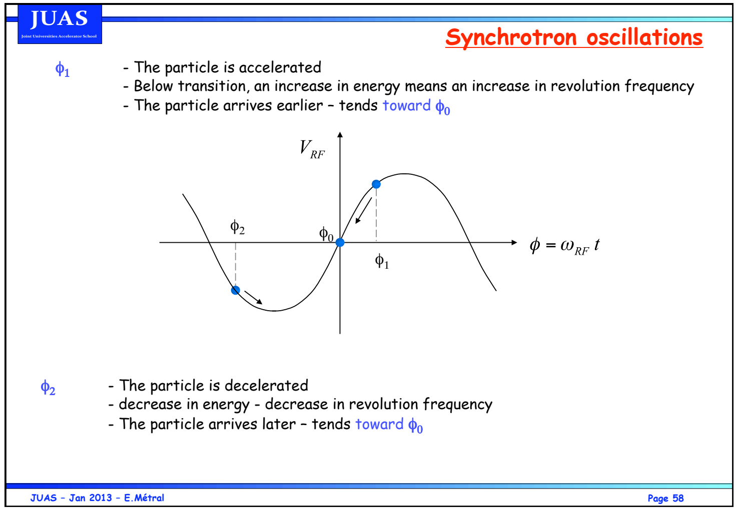
$$\omega = \frac{e\,B}{\gamma\, m_0} = \frac{\omega_{RF}}{h}$$

$$\Delta E = e\,\hat{V}_{RF}\,\sin\phi$$

In order to keep the resonant condition, the particle must keep a constant energy

The phase of the synchronous particle must therefore be $\phi_0 = 0$ (circular machines convention)

Let's see what happens for a particle with the same energy and a different phase (e.g., $\phi_1$)



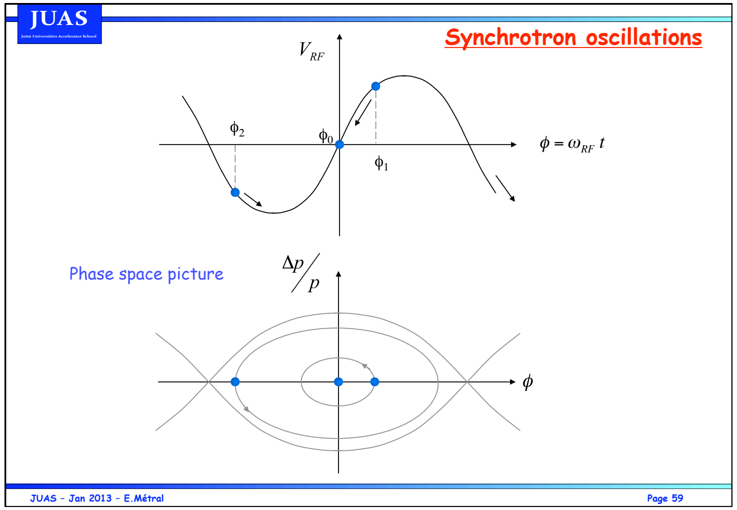
## Synchrotron oscillations

$\phi_1$

- The particle is accelerated
- Below transition, an increase in energy means an increase in revolution frequency
- The particle arrives earlier – tends toward $\phi_0$

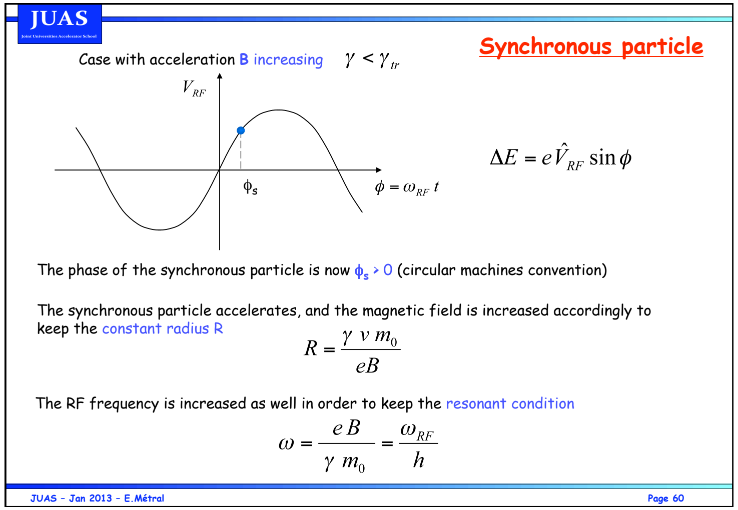
$\phi_2$

- The particle is decelerated
- decrease in energy - decrease in revolution frequency
- The particle arrives later – tends toward $\phi_0$



## Synchrotron oscillations

Phase space picture



## Synchronous particle

Case with acceleration B increasing $\gamma < \gamma_{tr}$

$$\Delta E = e\hat{V}_{RF}\,\sin\phi$$

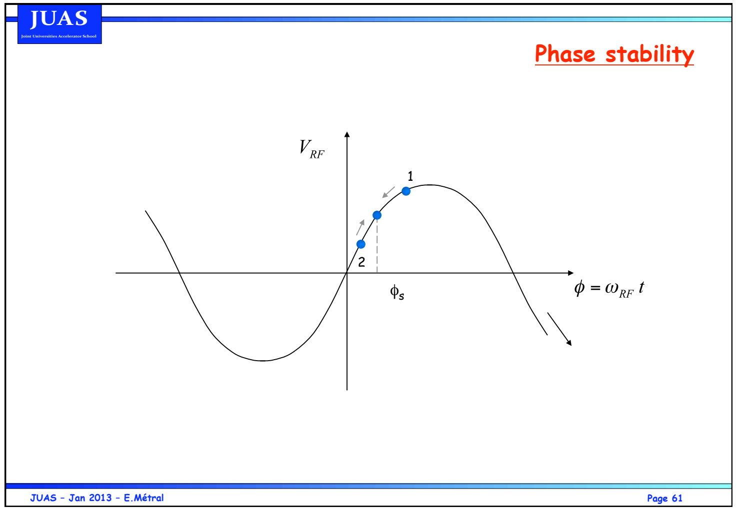
The phase of the synchronous particle is now $\phi_s > 0$ (circular machines convention)

The synchronous particle accelerates, and the magnetic field is increased accordingly to keep the constant radius R

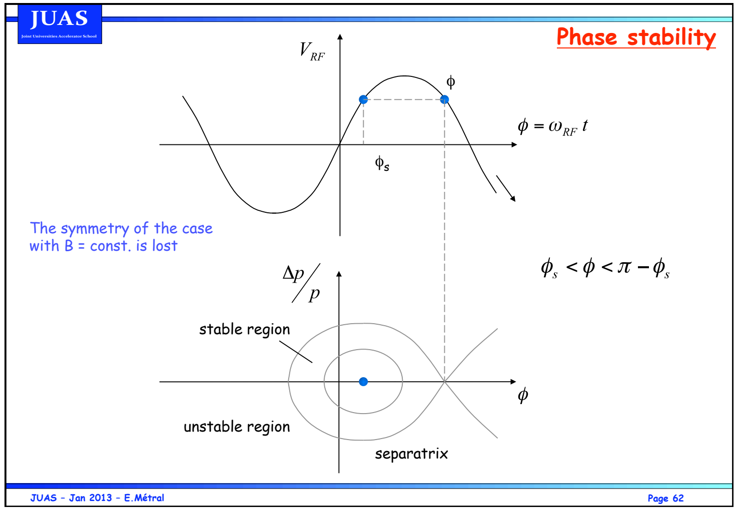
$$R = \frac{\gamma\; v\; m_0}{eB}$$

The RF frequency is increased as well in order to keep the resonant condition

$$\omega = \frac{e\,B}{\gamma\; m_0} = \frac{\omega_{RF}}{h}$$



## Phase stability



## Phase stability

The symmetry of the case with B = const. is lost

$$\phi_s < \phi < \pi - \phi_s$$

## LESSON IV

RF acceleration for synchronous particle

RF acceleration for non-synchronous particle

Small amplitude oscillations

Large amplitude oscillations - the RF bucket

## RF acceleration for synchronous particle - energy gain

Let's assume a synchronous particle with a given $\phi_s > 0$

We want to calculate its rate of acceleration, and the related rate of increase of B, f.

$$p = e\, B\, \rho$$

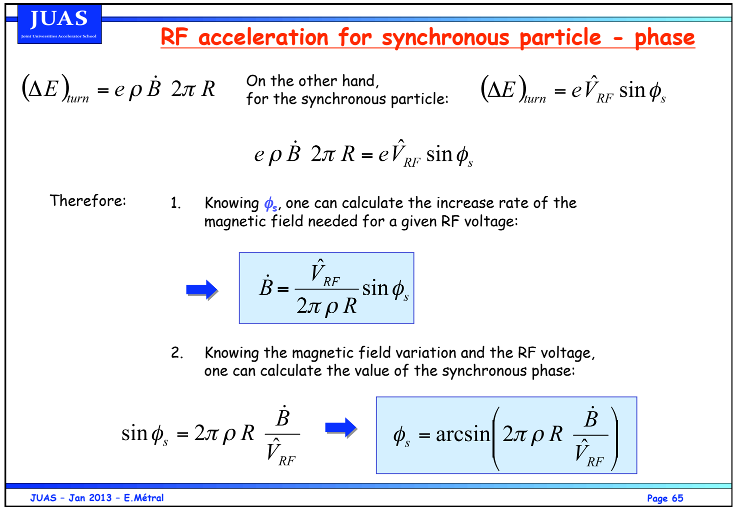
Want to keep $\rho$ = const

$$\Rightarrow \quad \frac{\mathrm{d}p}{\mathrm{d}t} = e\,\rho\,\frac{\mathrm{d}B}{\mathrm{d}t} = e\,\rho\,\dot{B}$$

Over one turn: $$(\Delta p)_{turn} = e\,\rho\,\dot{B}\, T_{rev} = e\,\rho\,\dot{B}\,\frac{2\pi R}{\beta c}$$

We know that (relativistic equations) : $$\Delta p = \frac{\Delta E}{\beta c}$$

$$\Rightarrow \quad \boxed{(\Delta E)_{turn} = e\,\rho\,\dot{B}\;2\pi\, R}$$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## RF acceleration for synchronous particle - phase

$$(\Delta E)_{turn} = e\,\rho\,\dot{B}\;2\pi\,R$$

On the other hand,
for the synchronous particle:

$$(\Delta E)_{turn} = e\hat{V}_{RF}\sin\phi_s$$

$$e\,\rho\,\dot{B}\;2\pi\,R = e\hat{V}_{RF}\sin\phi_s$$

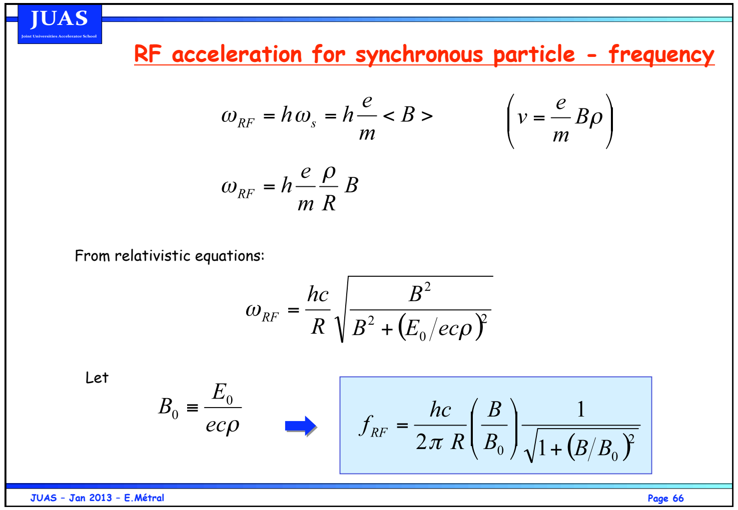
Therefore:

1. Knowing $\phi_s$, one can calculate the increase rate of the magnetic field needed for a given RF voltage:

$$\boxed{\dot{B} = \frac{\hat{V}_{RF}}{2\pi\,\rho\,R}\sin\phi_s}$$

2. Knowing the magnetic field variation and the RF voltage, one can calculate the value of the synchronous phase:

$$\sin\phi_s = 2\pi\,\rho\,R\,\frac{\dot{B}}{\hat{V}_{RF}} \Rightarrow \boxed{\phi_s = \arcsin\left(2\pi\,\rho\,R\,\frac{\dot{B}}{\hat{V}_{RF}}\right)}$$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## RF acceleration for synchronous particle - frequency

$$\omega_{RF} = h\omega_s = h\frac{e}{m} < B > \qquad \left(v = \frac{e}{m}B\rho\right)$$

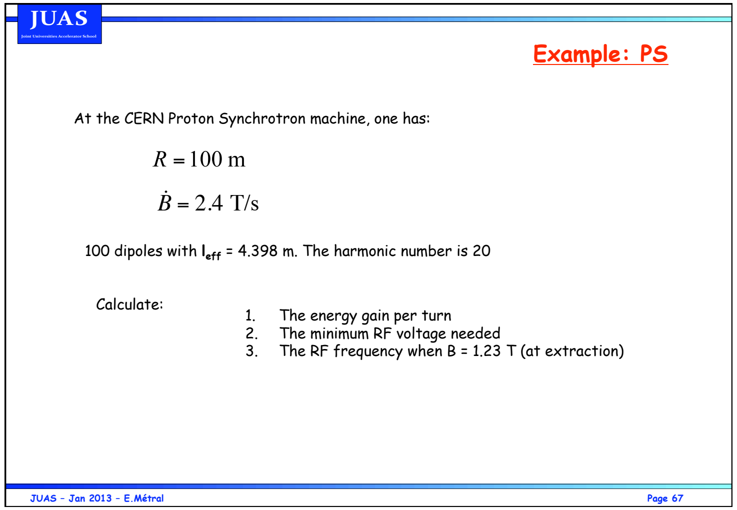
$$\omega_{RF} = h\frac{e}{m}\frac{\rho}{R}B$$

From relativistic equations:

$$\omega_{RF} = \frac{hc}{R}\sqrt{\frac{B^2}{B^2 + (E_0/ec\rho)^2}}$$

Let

$$B_0 \equiv \frac{E_0}{ec\rho} \Rightarrow \boxed{f_{RF} = \frac{hc}{2\pi\,R}\left(\frac{B}{B_0}\right)\frac{1}{\sqrt{1 + (B/B_0)^2}}}$$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## Example: PS

At the CERN Proton Synchrotron machine, one has:

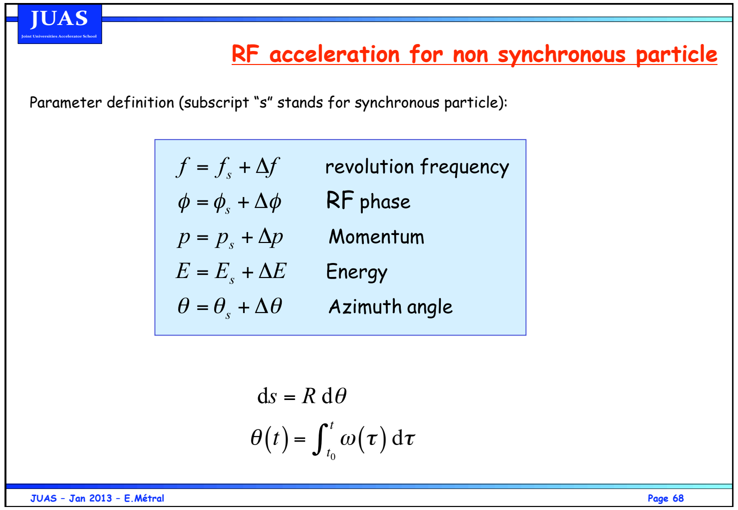
$$R = 100\ \mathrm{m}$$

$$\dot{B} = 2.4\ \mathrm{T/s}$$

100 dipoles with $l_{eff}$ = 4.398 m. The harmonic number is 20

Calculate:

1. The energy gain per turn
2. The minimum RF voltage needed
3. The RF frequency when B = 1.23 T (at extraction)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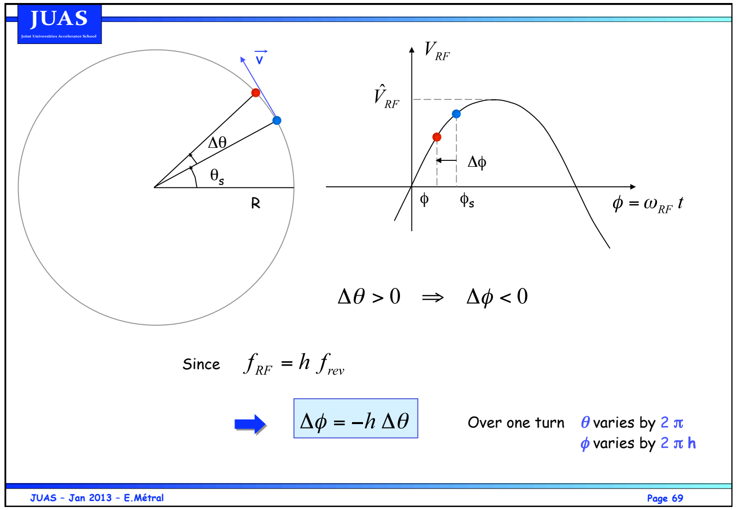
## RF acceleration for non synchronous particle

Parameter definition (subscript "s" stands for synchronous particle):

| | |
|---|---|
| $f = f_s + \Delta f$ | revolution frequency |
| $\phi = \phi_s + \Delta\phi$ | RF phase |
| $p = p_s + \Delta p$ | Momentum |
| $E = E_s + \Delta E$ | Energy |
| $\theta = \theta_s + \Delta\theta$ | Azimuth angle |

$$\mathrm{d}s = R\,\mathrm{d}\theta$$

$$\theta(t) = \int_{t_0}^{t}\omega(\tau)\,\mathrm{d}\tau$$



$$\Delta\theta > 0 \quad \Rightarrow \quad \Delta\phi < 0$$

Since $f_{RF} = h\, f_{rev}$

$$\boxed{\Delta\phi = -h\, \Delta\theta}$$

Over one turn $\theta$ varies by $2\,\pi$
$\phi$ varies by $2\,\pi\,h$

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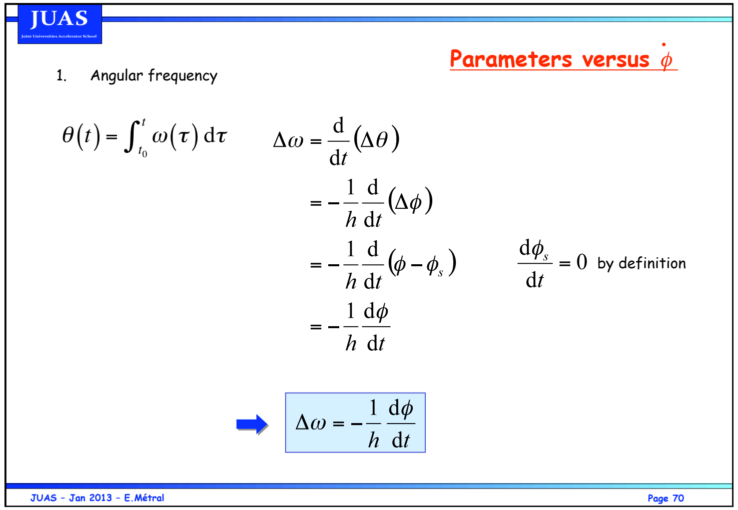

## Parameters versus $\dot{\phi}$

1. Angular frequency

$$\theta(t) = \int_{t_0}^{t} \omega(\tau)\, \mathrm{d}\tau$$

$$\Delta\omega = \frac{\mathrm{d}}{\mathrm{d}t}(\Delta\theta)$$
$$= -\frac{1}{h}\frac{\mathrm{d}}{\mathrm{d}t}(\Delta\phi)$$
$$= -\frac{1}{h}\frac{\mathrm{d}}{\mathrm{d}t}(\phi - \phi_s) \qquad \frac{\mathrm{d}\phi_s}{\mathrm{d}t} = 0 \text{ by definition}$$
$$= -\frac{1}{h}\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

$$\boxed{\Delta\omega = -\frac{1}{h}\frac{\mathrm{d}\phi}{\mathrm{d}t}}$$

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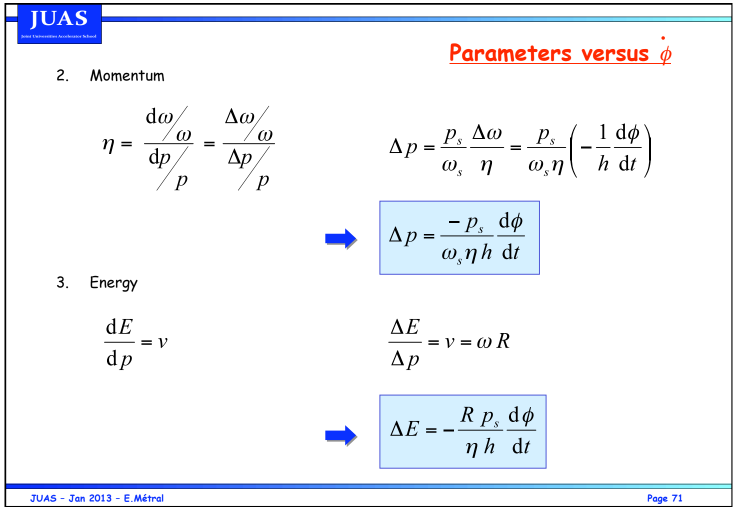

## Parameters versus $\dot{\phi}$

2. Momentum

$$\eta = \frac{\mathrm{d}\omega/\omega}{\mathrm{d}p/p} = \frac{\Delta\omega/\omega}{\Delta p/p} \qquad \Delta p = \frac{p_s}{\omega_s}\frac{\Delta\omega}{\eta} = \frac{p_s}{\omega_s\eta}\left(-\frac{1}{h}\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)$$

$$\boxed{\Delta p = \frac{-p_s}{\omega_s\eta\, h}\frac{\mathrm{d}\phi}{\mathrm{d}t}}$$

3. Energy

$$\frac{\mathrm{d}E}{\mathrm{d}p} = v \qquad \frac{\Delta E}{\Delta p} = v = \omega\, R$$

$$\boxed{\Delta E = -\frac{R\, p_s}{\eta\, h}\frac{\mathrm{d}\phi}{\mathrm{d}t}}$$

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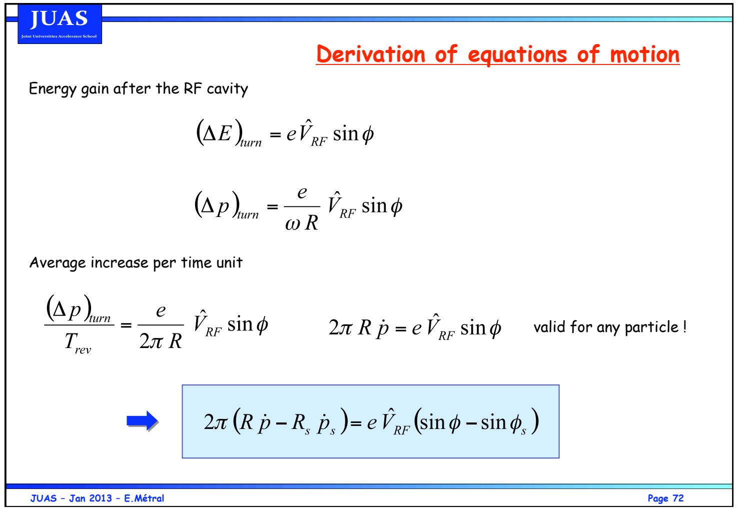


## Derivation of equations of motion

Energy gain after the RF cavity

$$(\Delta E)_{turn} = e\hat{V}_{RF}\sin\phi$$

$$(\Delta p)_{turn} = \frac{e}{\omega\, R}\,\hat{V}_{RF}\sin\phi$$

Average increase per time unit

$$\frac{(\Delta p)_{turn}}{T_{rev}} = \frac{e}{2\pi\, R}\,\hat{V}_{RF}\sin\phi \qquad 2\pi\, R\,\dot{p} = e\,\hat{V}_{RF}\sin\phi$$ valid for any particle !

$$\boxed{2\pi\left(R\,\dot{p} - R_s\,\dot{p}_s\right) = e\,\hat{V}_{RF}\left(\sin\phi - \sin\phi_s\right)}$$

## Derivation of equations of motion

After some development (see J. Le Duff, in Proceedings CAS 1992, CERN 94-01)

$$2\pi \frac{d}{dt}\left(\frac{\Delta E}{\omega_s}\right) = e\,\hat{V}_{RF}\left(\sin\phi - \sin\phi_s\right)$$

An approximated version of the above is

$$\boxed{\frac{\mathrm{d}(\Delta p)}{\mathrm{d}t} = \frac{e\,\hat{V}_{RF}}{2\pi\,R_s}\left(\sin\phi - \sin\phi_s\right)}$$

Which, together with the previously found equation

$$\boxed{\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{\omega_s\,\eta\,h}{p_s}\,\Delta p}$$

Describes the motion of the non-synchronous particle in the longitudinal phase space ( $\Delta p, \phi$ )

## Equations of motion I

$$\boxed{\begin{cases} \dfrac{\mathrm{d}(\Delta p)}{\mathrm{d}t} = A\left(\sin\phi - \sin\phi_s\right) \\ \dfrac{\mathrm{d}\phi}{\mathrm{d}t} = B\,\Delta p \end{cases}}$$

with $A = \dfrac{e\,\hat{V}_{RF}}{2\pi\,R_s}$

$$B = -\frac{\eta\,h}{p_s}\,\frac{\beta_s c}{R_s}$$

## Equations of motion II

1. First approximation - combining the two equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{B}\frac{\mathrm{d}\phi}{\mathrm{d}t}\right) - A\left(\sin\phi - \sin\phi_s\right) = 0$$

We assume that A and B <u>change very slowly</u> compared to the variable $\Delta\phi = \phi - \phi_s$

$$\Rightarrow \boxed{\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} + \frac{\Omega_s^{\,2}}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0}$$

with $\dfrac{\Omega_s^{\,2}}{\cos\phi_s} = -A\,B$

We can also define: $\Omega_0^{\,2} = \dfrac{\Omega_s^{\,2}}{\cos\phi_s} = \dfrac{e\,\hat{V}_{RF}\,\eta\,h\,c^2}{2\pi\,R_s^{\,2}E_s}$

## Small amplitude oscillations

2. Second approximation

$$\sin\phi = \sin(\phi_s + \Delta\phi)$$
$$= \sin\phi_s \cos\Delta\phi + \cos\phi_s \sin\Delta\phi$$

$\Delta\phi$ small $\Rightarrow$ $\sin\phi \cong \sin\phi_s + \cos\phi_s\,\Delta\phi$

$\dfrac{\mathrm{d}\phi_s}{\mathrm{d}t} = 0$ $\Rightarrow$ $\dfrac{\mathrm{d}^2\phi}{\mathrm{d}t^2} = \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\phi_s + \Delta\phi\right) = \dfrac{\mathrm{d}^2\Delta\phi}{\mathrm{d}t^2}$

by definition

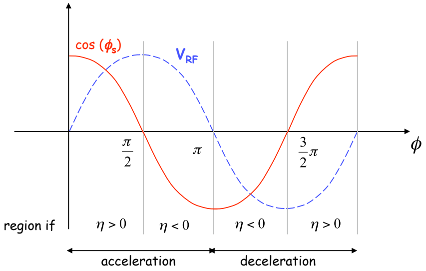
$$\Rightarrow \boxed{\frac{\mathrm{d}^2\Delta\phi}{\mathrm{d}t^2} + \Omega_s^{\,2}\,\Delta\phi = 0}$$

Harmonic oscillator !

## Stability condition for $\phi_s$

Stability is obtained when the angular frequency of the oscillator, $\Omega_s^{\ 2}$ is real positive:

$$\boxed{\Omega_s^{\ 2} = \frac{e\,\hat{V}_{RF}\,\eta\,h\,c^2}{2\pi\,R_s^{\ 2}E_s}\cos\phi_s} \;\Rightarrow\; \Omega_s^{\ 2} > 0 \;\Leftrightarrow\; \boxed{\eta\cos\phi_s > 0}$$



Stable in the region if

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## Small amplitude oscillations - orbits

For $\eta\cos\phi_s > 0$ the motion around the synchronous particle is a stable oscillation:

$$\boxed{\begin{cases} \Delta\phi = \Delta\phi_{\max}\sin\left(\Omega_s t + \phi_0\right) \\ \Delta p = \Delta p_{\max}\cos\left(\Omega_s t + \phi_0\right) \end{cases}}$$

with $\Delta p_{\max} = \frac{\Omega_s}{B}\Delta\phi_{\max}$

---

## Lepton machines

e+, e-

$$\boxed{\beta \cong 1 \quad , \quad \gamma \text{ large} \quad , \quad \eta \cong -\alpha_p}$$

$$\omega_s \cong \frac{c}{R_s} \quad , \quad p_s \cong \frac{E_s}{c} \quad \Rightarrow \quad \boxed{\Omega_s = \frac{c}{R_s}\left\{-\frac{e\,\hat{V}_{RF}\,\alpha_p\,h}{2\pi\,E_s}\cos\phi_s\right\}^{1/2}}$$

Number of synchrotron oscillations per turn:

$$Q_s = \frac{\Omega_s}{\omega_s} = \left\{-\frac{e\,\hat{V}_{RF}\,\alpha_p\,h}{2\pi\,E_s}\cos\phi_s\right\}^{1/2}$$

"synchrotron tune"

N.B: in these machines, the RF frequency does not change

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## Large amplitude oscillations

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

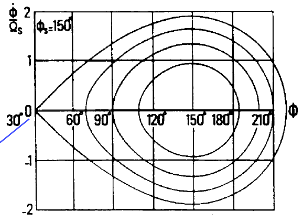
Multiplying by $\dot{\phi}$
and integrating

$$\boxed{\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = cte}$$

Constant of motion

here $\dot{\phi} = 0$

$\phi = \pi - \phi_s$



Synchronous phase 150°

Equation of the separatrix

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\ \sin\phi_s\right) = -\frac{\Omega_s^2}{\cos\phi_s}\left[\cos\left(\pi - \phi_s\right) + \left(\pi - \phi_s\right)\sin\phi_s\right]$$

JUAS

## Energy diagram

"total energy"

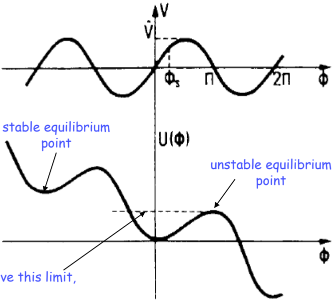
$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = cte$$

"kinetic energy"

"potential energy U "

$$\frac{d^2\phi}{dt^2} = F(\phi)$$

$$F(\phi) = -\frac{\partial U}{\partial \phi}$$



stable equilibrium point

unstable equilibrium point

If the total energy is above this limit, the motion is unbounded

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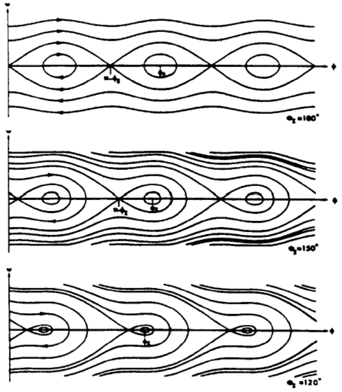
## Phase space trajectories



$\gamma > \gamma_{tr}$

Phase space trajectories for different synchronous phases