Beam Transverse Oscillations with strong space charge

A. Burov

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DIETER MÖHL – PROMINENT SCIENTIST AND WONDERFUL TEACHER

Every time I saw him, I was truly enraptured by his very special aura – the aura of goodness, intelligence and love, irradiated from the very core of his soul…

Content

Möhl – Schönauer Equation

- In 1974, D. Möhl and H. Schönauer suggested to describe coasting beam oscillations by the following equation:

$$\frac{d^2x_i}{dt^2}+\Omega_i^{\,2}\underset{\substack{\nearrow\\ \text{lattice}}}{Q_i^{\,2}}x_i+2\Omega_0^{\,2}Q_0\left[\underset{\substack{\nearrow\\ \text{wake}}}{Q_c}\,\bar{x}+Q_{sc}\underset{\substack{\uparrow\\ \text{space charge}}}{\left(x_i-\bar{x}\right)}\right]=0\,.$$

$$Q_{sc}=Q_{sc}(\underset{\substack{\uparrow\\ \text{action}}}{\mathbf{J}_i});\qquad Q_{sc}(0)=-\frac{N r_0}{4\pi\beta^2\gamma^3\varepsilon_{\mathrm{rms}}}\,.$$

Coherent tune shift, in a frequency domain:

$$Q_c=-i\,\frac{N r_0\beta_x}{\gamma C}\,\frac{Z_x}{Z_0}\,;\qquad Z_0=\frac{4\pi}{c}=377\,\mathrm{Ohm}$$

Space charge term

$$\frac{d^2 x_i}{dt^2} + \Omega_i{}^2 Q_i{}^2 x_i + 2\Omega_0{}^2 Q_0 \left[Q_c \bar{x} + Q_{sc} \left(x_i - \bar{x} \right) \right] = 0 \, .$$

- The only non-trivial term in this equation is the space charge force:

Space charge term

$$\frac{d^2x_i}{dt^2}+\Omega_i{}^2Q_i{}^2x_i+2\Omega_0{}^2Q_0\left[Q_c\bar{x}+Q_{sc}\left(x_i-\bar{x}\right)\right]=0\,.$$

- The only non-trivial term in this equation is the space charge force:

$$Q_{sc}\left(x_i-\bar{x}\right)$$

Space charge term

$$\frac{d^2 x_i}{dt^2} + \Omega_i{}^2 Q_i{}^2 x_i + 2\Omega_0{}^2 Q_0 \left[Q_c \bar{x} + Q_{sc} \left(x_i - \bar{x} \right) \right] = 0 .$$

- The only non-trivial term in this equation is the space charge force:

$$Q_{sc} \left(x_i - \bar{x} \right)$$

- Why non-linear space charge forces are described by a linear term? Does it mean that it is valid only for K-V (constant density profile) distribution?

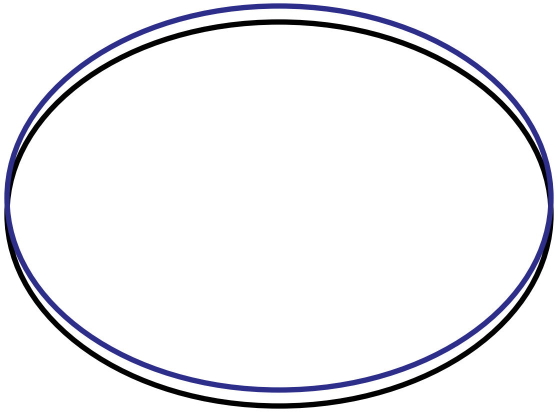
- No, it is valid for any beam profile. The reason is following:

Comments to Möhl-Schönauer Equation (MSE)

Space charge term: $Q_{sc}(\mathbf{J}_\perp)\left(x_i - \bar{x}\right)$

- The single-particle motion consists of 2 parts:
  - free oscillations (typically with beam size amplitude, or $J_\perp \simeq \varepsilon_{\mathrm{rms}}$)
  - driven by the coherent offset oscillations (much smaller than the beam size).

- The equation describes the driven oscillations only, so it results from linearization of the original non-linear space charge term over infinitesimally small coherent motion, and averaging over the betatron phases of the free incoherent oscillations.



MSE as a unique choice

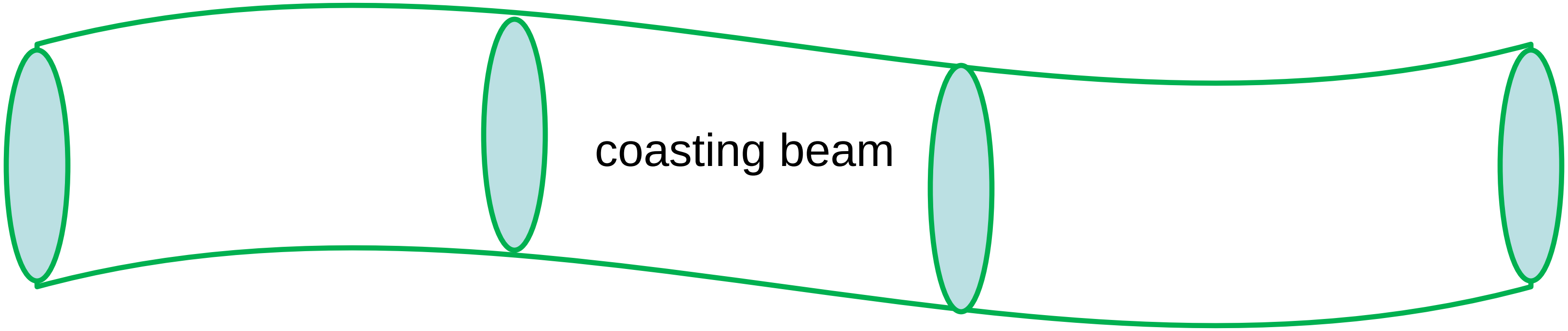
- This equation is a unique possibility for
  - A linear equation (driven oscillations are small!)
  - With time-independent coefficients (= rigid slice approximation)
  - With the given tunes and incoherent SC tune shifts $Q_{sc}(\mathbf{J}_\perp)$
  - Space charge force $\propto (x_i - \overline{x})$

MSE is based on a rigid-slice approximation

$$\frac{d^2 x_i}{dt^2} + \Omega_i{}^2 Q_i{}^2 x_i + 2\Omega_0{}^2 Q_0 \left[Q_c \bar{x} + Q_{sc} \left(x_i - \bar{x} \right) \right] = 0 .$$

lattice wake space charge

- The assumption for the rigid slice approximation is that a core of any beam slice moves as a whole, so there is no inner motion in it.



- This is a good approximation, when the space charge is strong enough (Burov & Lebedev, 2008):

$$| Q_{sc} | >> \Delta Q_i .$$

- Otherwise, the beam shape oscillates as well, and MSE is not necessarily valid.

Coasting beam: Landau damping (Burov, Lebedev, 2008):

$$\boxed{\Lambda = -\pi \left\langle \Delta Q_{sep} \right\rangle \int \Delta Q_{sep} f_x J_x \; \delta(\Delta Q_l + Q_{sc} - \mathrm{Re}\,\nu_c) d\Gamma,} \qquad f_x \equiv \partial f / \partial J_x \; ;$$

$$\Delta Q_{sep} \equiv \mathrm{Re}\,\Delta Q_c - Q_{sc}(J_x, J_y)\,, \qquad \left\langle \Delta Q_{sep} \right\rangle \equiv -\left(\int \frac{f_x J_x \; d\Gamma}{\Delta Q_{sep}} \right)^{-1} ;$$

$$\mathrm{Re}\,\nu_c = \mathrm{Re}\,\Delta Q_c + \delta Q^{(1)} + \delta Q^{(2)} \,,$$

$$\mathrm{Im}\,\nu_c = \mathrm{Im}\,\Delta Q_c - \Lambda$$

$$\delta Q^{(1)} = -\left\langle \Delta Q_{sep} \right\rangle \int \frac{\Delta Q_l \, f_x J_x \; d\Gamma}{\Delta Q_{sep}} \,,$$

$$\delta Q^{(2)} = -\left\langle \Delta Q_{sep} \right\rangle \int \frac{\Delta Q_l^2 \, f_x J_x \; d\Gamma}{\Delta Q_{sep}^2} \,.$$

Landau Damping

- Landau damping is proportional to the phase space density of the resonance particle:

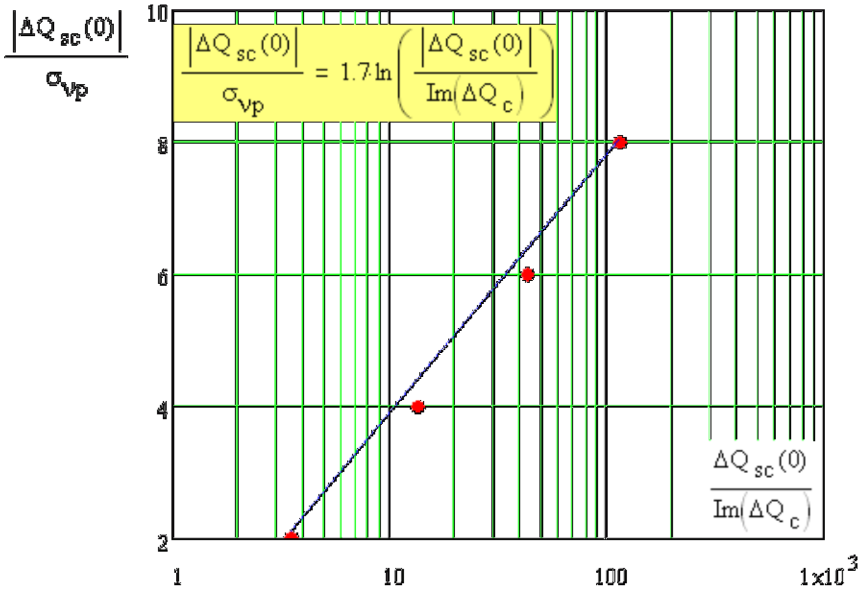
$$\Lambda = -\pi \left\langle \Delta Q_{sep} \right\rangle \int \Delta Q_{sep} f_x J_x \ \delta(\Delta Q_l + Q_{sc} - \mathrm{Re}\,\nu_c) d\Gamma$$

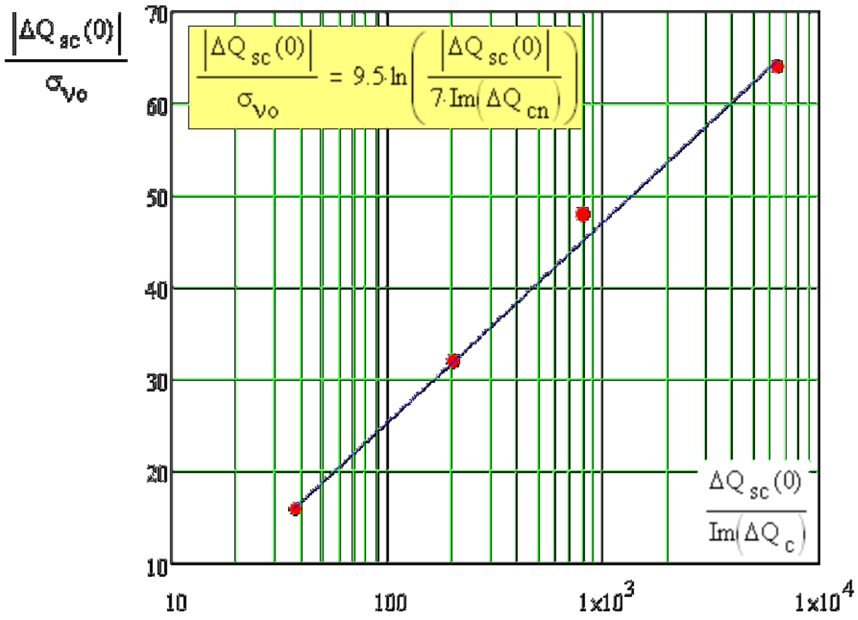
- Thus, with strong space charge, it is determined by the distribution tails.

- Typically, the far tails are neither predictable nor measurable, and not even reproducible in many cases. That is why, at strong space charge, the stability thresholds are predictable and reproducible with poor accuracy, like factor of 1.5-2.

Coasting Beam Thresholds (Burov, Lebedev, 2008)

- Thresholds are determined by $\Lambda = \mathrm{Im}\,\Delta Q_c$. For a round Gaussian beam:





Chromatic threshold

Octupole threshold

$$\sigma_{vp} \equiv |\, \xi - n\eta \,| \left(\frac{\delta p}{p}\right)_{\mathrm{rms}}$$

$$\Delta Q_l = \left(\xi - n\eta\right)\hat{p} \equiv \sigma_{vp}\,\hat{p} / \sigma_p$$

$$\Delta Q_i = \sigma_{vo}(J_x + J_y)/2\varepsilon \;\; > 0\,;$$

$$\langle J_x \rangle = \langle J_y \rangle = \varepsilon$$

Direct application to bunched beams

- Coasting beam results sometimes can be applied to bunched beams.
    - For short-wavelength and fast oscillations (much shorter than the bunch length, much faster than the synchrotron frequency) – the so-called microwave instability;
    - For multi-bunched beam, at zero chromaticity and long wavelength (much longer than the bunch spacing)

Bunched Beams (no SC – Sacherer theory)

- Jeans-Vlasov Equation:

$$\frac{\partial \psi}{\partial t} + \omega_b \frac{\partial \psi}{\partial \phi_x} + \omega_s \frac{\partial \psi}{\partial \phi_s} + F_x \frac{\partial \psi}{\partial p_x} = 0;$$

$$F_x(z) = \int W(z - z')\bar{x}(z')dz' = \int W(z - z')x' \psi(\Gamma')d\Gamma'$$

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$$F_x(z) = \int W(z - z')\bar{x}(z')dz' = \int W(z - z')x'\psi(\Gamma')d\Gamma'$$

$$\psi = f_0(J_x)g_0(J_s) + \tilde{f}(J_x, \phi_x)\tilde{g}(J_s, \phi_s)e^{-i\Omega t}$$

Linearization

$$\tilde{f}(J_x, \phi_x) = -D\sqrt{2J_x}\, f_0^{'}(J_x)e^{i\phi_x}$$

Dipole motion

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$$\frac{\partial\psi}{\partial t}+\omega_b\frac{\partial\psi}{\partial\phi_x}+\omega_s\frac{\partial\psi}{\partial\phi_s}+F_x\frac{\partial\psi}{\partial p_x}=0;$$

$$F_x(z)=\int W(z-z')\bar{x}(z')dz'=\int W(z-z')x'\psi(\Gamma')d\Gamma'$$

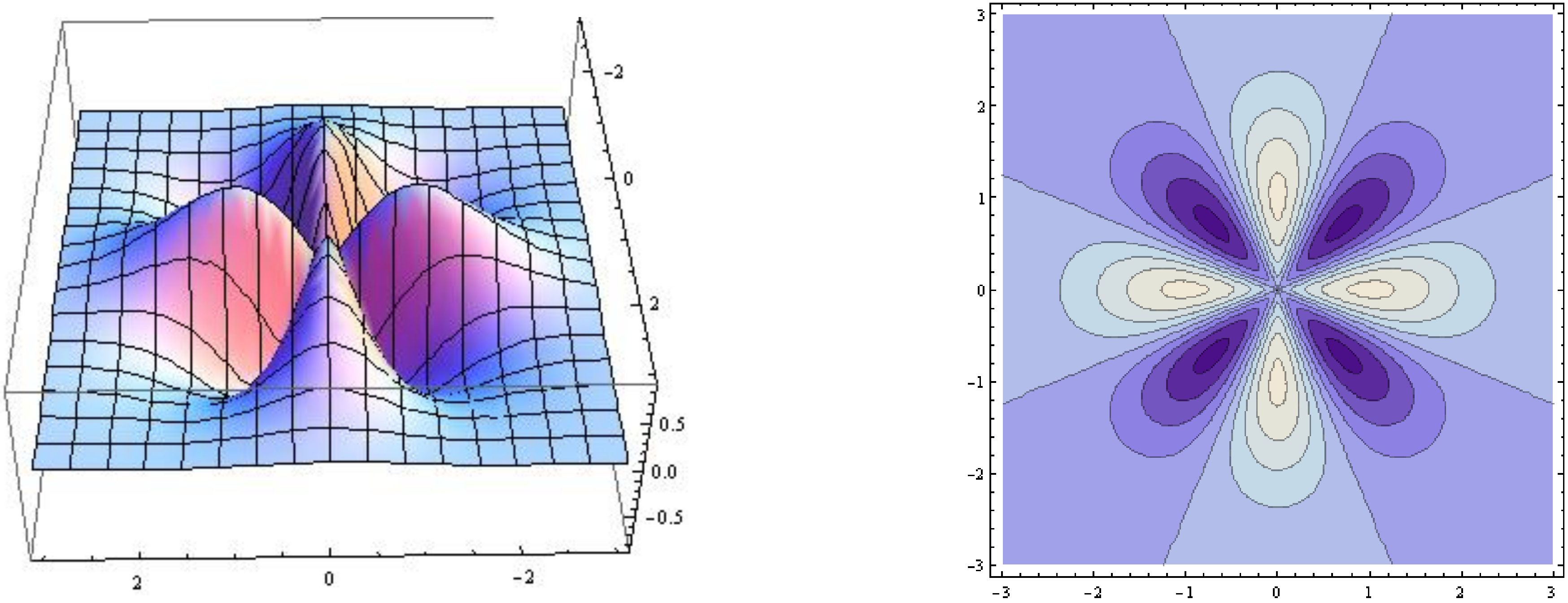
$$\psi = f_0(J_x)g_0(J_s)+\tilde{f}(J_x,\phi_x)\tilde{g}(J_s,\phi_s)e^{-i\Omega t}$$

$$\tilde{f}(J_x,\phi_x)=-D\sqrt{2J_x}\,f_0{}'(J_x)e^{i\phi_x}$$

Thus, the only function to be found is $\tilde{g}(J_s,\phi_s)=R_n(J_s)e^{im\phi_s}\equiv\tilde{g}_{nm}(J_s,\phi_s),$

describing the longitudinal phase space variation of the transverse offset.

Typical solution for the offset modulation $\tilde{g}_{n,m}(J_s,\phi_s)$



3D and Contour plots for $\tilde{g}_{1,4}(J_s,\phi_s)$ (example)

$$Q_c = -i\frac{Nr_0\beta_x}{\gamma C}\sum_{p=-\infty}^{\infty} Z(\omega')J_m^2(\omega'\tau_b-\chi)$$

coherent tune shift (air-bag model)

$$\omega' = \omega_b + p\omega_0 \qquad \chi = \frac{\xi\sigma_p}{Q_s}$$

Condition for Landau damping: $|m|\Delta Q_s + \Delta Q_b \geq C|Q_c|; \quad C \sim 0.2 - 0.5$

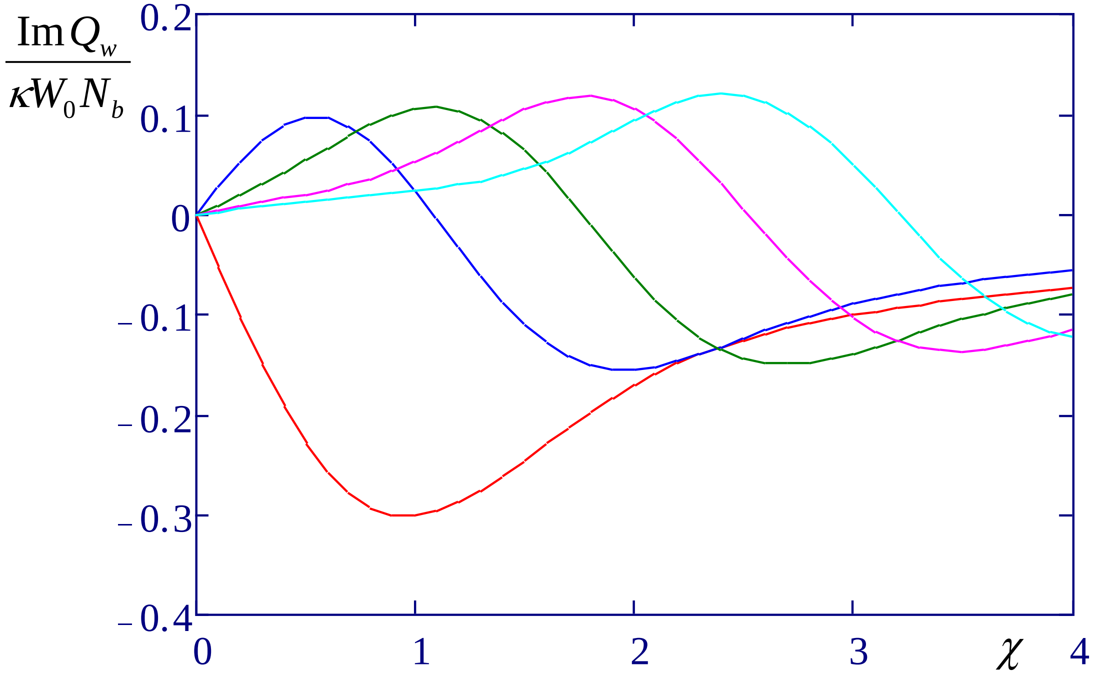
Wake+ : weak head-tail

- Wake yields its coherent tune shift for every mode. If it is small $|Q_c| << |\nu_k - \nu_{k+1}|$ (*weak head-tail*), it can be calculated as a diagonal matrix element (similar to QM):

$$Q_c = \langle k | \hat{W} | k \rangle$$

- The result depends on the chromaticity ξ through the head-tail phase

$$\chi = \frac{\xi \sigma_s}{R\eta} = \frac{\xi \sigma_p}{Q_s} \mathrm{sign}(\eta)$$



Growth rates for the constant wake for the first 5 modes.

All of them are odd functions of the chromaticity.

$$\sum_k \mathrm{Im} Q_c = 0$$

Wake+ : strong head-tail

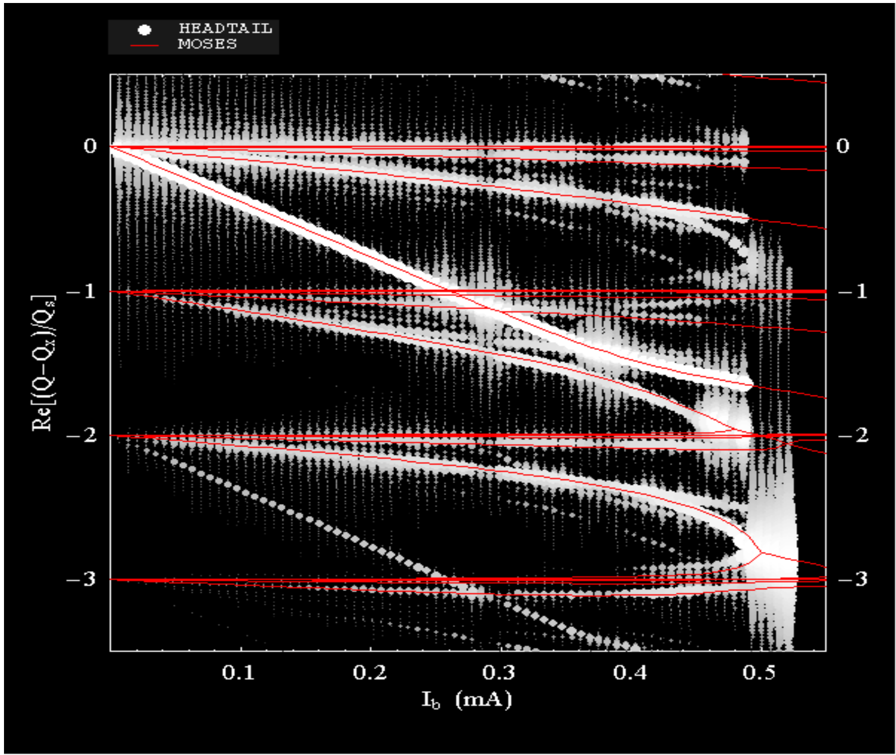
- In the opposite case $|Q_c| \geq |\nu_k - \nu_{k+1}|$, the nearest modes are coupled by the wake, leading to instability even at $\chi = 0$. This is called strong head-tail or transverse mode-coupling instability, TMCI.

- For electron machines (no space charge), typically modes k=0 and k=-1 are coupled sooner than other, since $Q_c = \langle 0|\hat{W}|0\rangle$ is maximal, and the modes are shifted down.

- After coupling of modes k and k+1:

$$\mathrm{Im}\nu_k = -\mathrm{Im}\nu_{k+1} \cong Q_s$$

B. Salvant et al, HB2008, TMCI simulations, no space charge



Möhl-Schönauer Equation (MSE) for bunched beams

Using slow betatron amplitudes $x_i(\theta)$,

$$X_i(\theta) = \exp(-iQ_b\theta)x_i(\theta)$$

the MSE writes as

$$\dot{x}_i(\theta) = iQ(\tau_i(\theta))\left[x_i(\theta) - \bar{x}(\theta,\tau_i(\theta))\right] - i\varsigma \mathrm{v}_i(\theta) x_i(\theta) - i\kappa\hat{\mathbf{W}}\bar{x}$$

space charge ↑ chromaticity ↑ wake ←

$\dot{x}_i = dx_i / d\theta$, $\zeta = -\xi / \eta$, $\eta = \gamma_t^{-2} - \gamma^{-2}$, $\xi = dQ_b / d(\Delta p / p)$, $\mathrm{v}_i(\theta) = \dot{\tau}_i(\theta)$

Here θ and τ are time and distance along the bunch, both in angle units.

MSE for bunched beam – cont.

- After a substitution $x_i(\theta) = y_i(\theta)\exp(-i\zeta\tau_i(\theta))$ with a new variable y, the chromatic term disappears from the equation, going instead into the wake term:

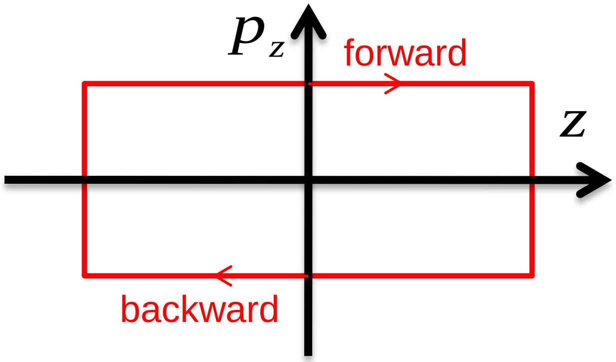
$$\dot{y}_i(\theta) = iQ(\tau_i(\theta))\left[y_i(\theta) - \bar{y}(\theta, \tau_i(\theta))\right] - i\kappa\hat{\mathbf{W}}\bar{y}$$

$$\kappa = \frac{r_0 R}{4\pi\beta^2\gamma Q_b}; \qquad \hat{\mathbf{W}}\bar{y} = \int_{\tau}^{\infty} W(\tau - s)\exp(i\zeta(\tau - s))\rho(s)\bar{y}(s)ds.$$

- Thus, for no-wake case, the bunched beam modes do not depend on the chromaticity, except the head-tail modulation $\propto \exp(-i\zeta\tau)$. Neither Landau damping, nor eigenfrequencies depend on the chromaticity for $W=0$.

Bunched Beam: Square Well Model

- For a square potential well and KV transverse distribution, the head-tail modes with space charge were described by Mike Blaskiewicz (1998).
- For the air-bag distribution, there are two particle fluxes in the synchrotron phase space:



- Since $Q_{sc} = \mathrm{const}$, MSE is easier to solve in this case.

Coherent tunes, no wake

general result:

$$\nu_{k\pm} = \nu_b - \frac{Q_{sc}}{2} \pm \sqrt{\frac{Q_{sc}^2}{4} + k^2 Q_s^2} \quad ; \quad k = 0, 1, 2...$$

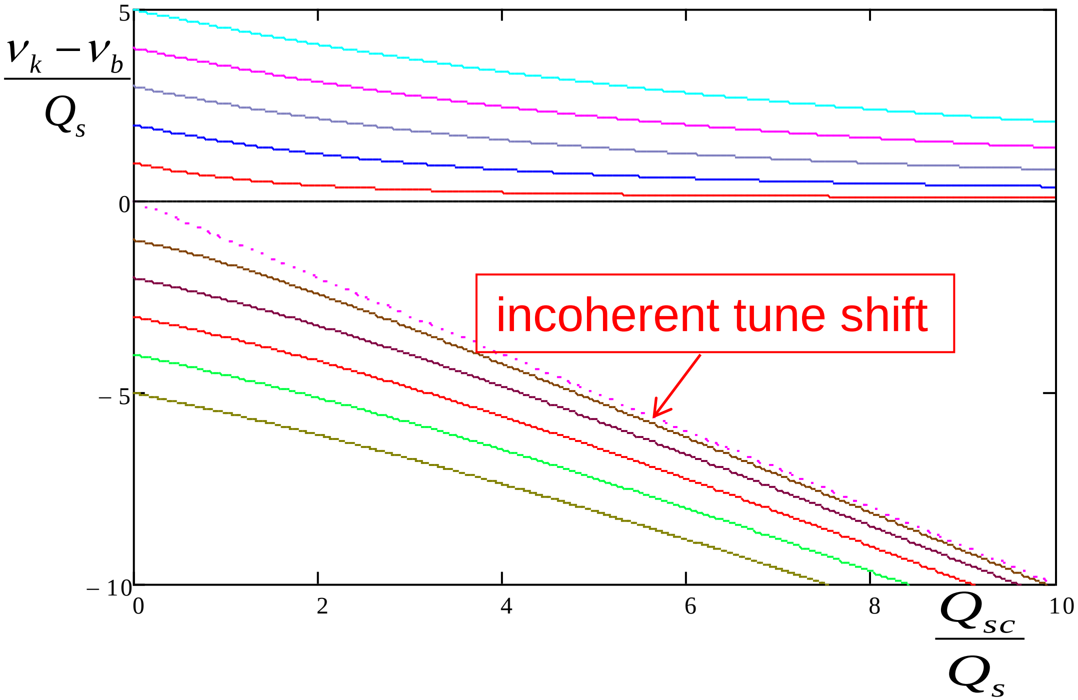
high space charge:

$$\nu_{k+} = \nu_b + \frac{k^2 Q_s^2}{Q_{sc}} \quad ; \quad k = 0,1,2...$$

$$\nu_{k-} = \nu_b - Q_{sc} - \frac{k^2 Q_s^2}{Q_{sc}}$$

small space charge:

$$\nu_k = \nu_b + kQ_s - \frac{Q_{sc}}{2} \quad ; \quad k = \pm 1, \pm 2...$$



incoherent tune shift

At high space charge, the + and - fluxes oscillate in phase for the positive modes and out of phase for the negative modes.

Only positive modes matter for high space charge.

Landau damping

- The wake-driven growth rate $\mathrm{Im}\,Q_c$ can be compensated by Landau damping Λ_L. Stability requires

$$\Lambda_L \geq \mathrm{Im}\,Q_c$$

- For the square-well model, the Landau damping is identical to the coasting beam case, except the chromaticity is dropped for the bunched case $\xi + n\eta \rightarrow n\eta$ (Blaskiewicz, 2003)

- As a result, the condition for the resonant particles to exist (LD condition) for the square well case is

$$k\Delta Q_s \geq C\max(Q_{sc}, |Q_c|)\ ; \quad C \sim 0.2 - 0.5$$

- Thus, for the strong space charge case, $Q_{sc} >> k\Delta Q_s$, there is no Landau damping for square potential well.

- What about Landau damping for other bucket shapes?

General solution of MSE

- After a substitution $x_i(\theta) = y_i(\theta)\exp(-i\zeta\tau_i(\theta))$ with a new variable y , the chromatic term disappears from the equation, going instead into the wake term:

$$\dot{y}_i(\theta) = iQ(\tau_i(\theta))\left[y_i(\theta) - \bar{y}(\theta, \tau_i(\theta))\right] - i\kappa\hat{\mathbf{W}}\bar{y}$$

$$\kappa = \frac{r_0 R}{4\pi\beta^2\gamma Q_b}\;; \qquad \hat{\mathbf{W}}\bar{y} = \int_{\tau}^{\infty} W(\tau - s)\exp(i\zeta(\tau - s))\rho(s)\bar{y}(s)ds\,.$$

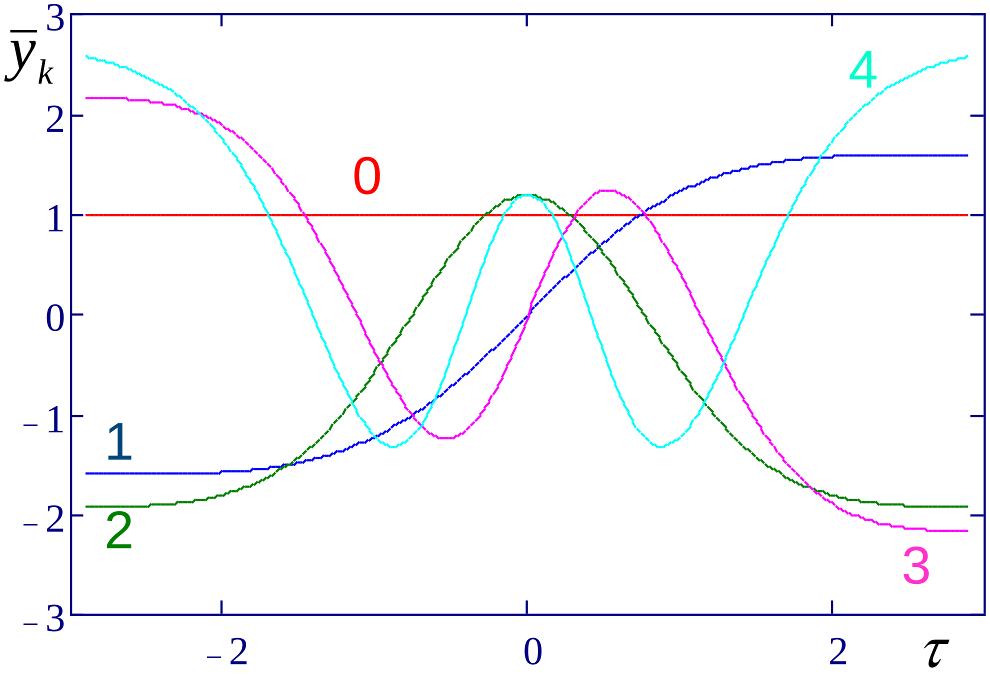
From here, a 2nd order ordinary IDE follows (A. Burov, 2009):

$$\frac{d}{d\tau}\left(u^2\frac{d\bar{y}}{d\tau}\right) + \nu_k Q_{sc}\bar{y} = \kappa Q_{sc}\hat{W}\bar{y}\,; \qquad \bar{y}'(\pm\infty) = 0$$

$$u^2 \equiv \frac{\int_{-\infty}^{\infty} f(\mathrm{v},\tau)d\mathrm{v}}{\int_{-\infty}^{\infty} \mathrm{v}^2 f(\mathrm{v},\tau)d\mathrm{v}}\,.$$

$Q_{sc}(\tau)$ is cross-section averaged.

No-wake (space charge only) modes, Gaussian bunch



These modes do not look too different from no-space charge case.
The only significant difference is that for strong space charge, modes are counted by a single integer, while conventional zero-space-charge modes require 2 integers: for azimuthal and radial numbers (due to their possible variations along synchrotron phase and action).

Landau damping for strong space charge (SC)

- Landau damping results from energy transfer from coherent modes to incoherent motion of resonant particles.

- For strong space charge $Q_{sc} >> Q_s$, particles and modes are tune-separated by Q_{sc} , so their resonance may seem to be impossible.

- However, this is not generally correct, since the SC tune shift goes to 0 at the bunch tails, where this resonance may happen.

- The higher is SC, the further to the tails it happens. Thus, space charge strongly suppresses Landau Damping.

Landau damping – results

- According to (Burov, 2009), for Gaussian bunch LD is estimated as:

$$\Lambda_k \cong 0.1 k^4 Q_s \left(\frac{Q_s}{Q_{sc}} \right)^3$$

(the numerical factor ~ 0.1 – best fit of V. Kornilov, GSI, to his numerical simulations, 2010)

- Note: mode k=0 is not damped at all. The damper is needed for it and may be for a few higher modes.

Vanishing TMCI

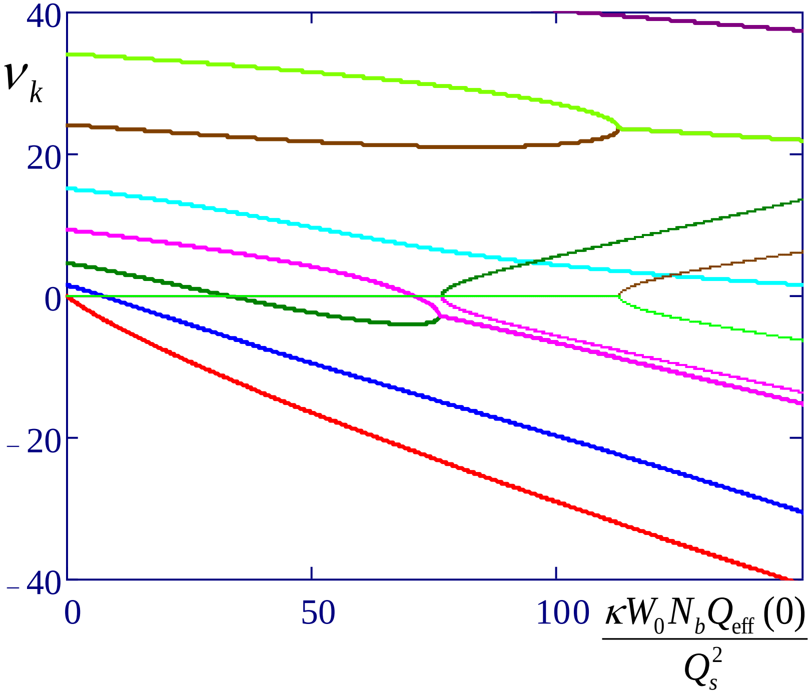
- While the conventional head-tail modes are numbered by integers,

$$\nu_k = kQ_s \, , k = 0, \pm 1, \pm 2, ...$$

  the space charge modes are numbered by natural numbers:

$$\nu_k \propto k^2 \, .$$

- This is a structural difference, leading to significant increase of the transverse mode coupling instability: the most affected lowest mode has no neighbor from below.

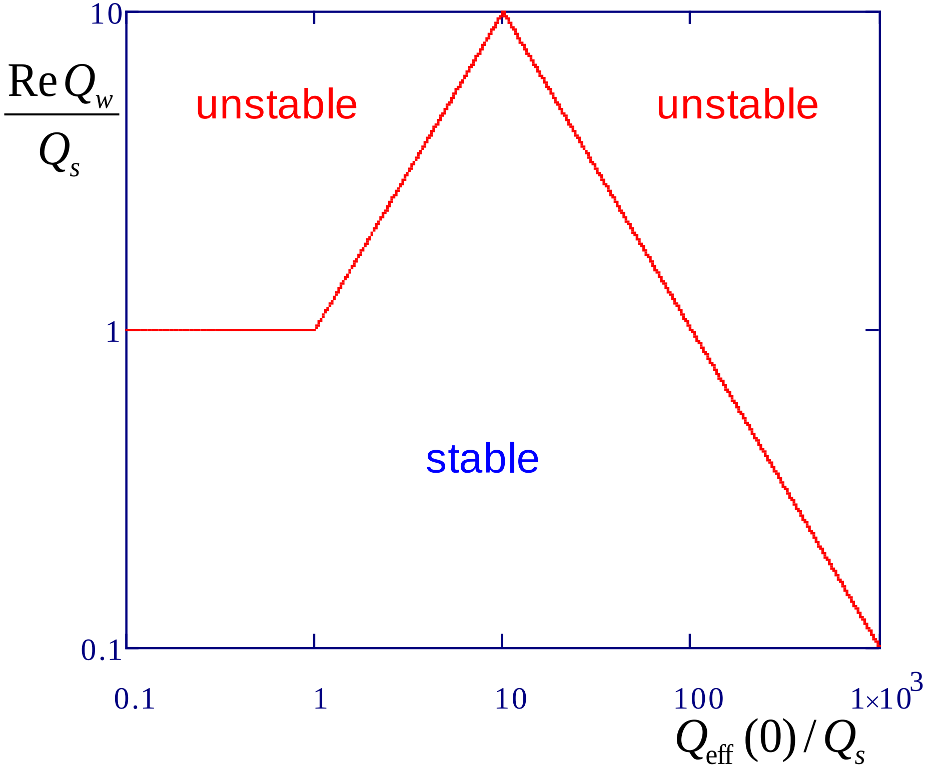


Coherent tunes of the Gaussian bunch for zero chromaticity and constant wake versus the wake amplitude.

Note high value of the TMCI threshold.

TMCI threshold for arbitrary space charge

A schematic behavior of the TMCI threshold for the coherent tune shift versus the space charge tune shift.



Summary

- Theory of transverse coherent oscillations of beams is approximately established now – both for coasting and bunched cases, with any space charge.

- The remaining problem – intermediate SC $Q_{sc} \simeq kQ_s$ (TMCI condition?)

- For strong space charge, stability threshold is determined by far tails of distribution – hardly visible and not always reproducible. This limits the prediction accuracy – both for coasting and bunched cases.

- For bunched beam with strong space charge, recent simulations (O. Boine-Frankenheim and V. Kornilov, GSI) so far confirm the theory. More simulation details are needed and expected. No observations to verify the theory are yet available.

- For TMCI, moderate SC improves stability, but too high SC makes the bunch more unstable.