

Beam Physics with Coupled Optics

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Equations of Motion

- For a linear Hamiltonian system with two degrees of freedom, equations of motion can be written in the following matrix form:

$$\frac{d\mathbf{x}}{ds} = \mathbf{U}\mathbf{H}\mathbf{x}, \quad \mathbf{U} = \begin{bmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \end{bmatrix}, \quad \mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\mathbf{H}(s)_{ik} = \partial^2 H / \partial x_i \partial x_k \quad \text{Hessian matrix of the Hamiltonian}$$

$$\mathbf{x} = (x, \theta_x, y, \theta_y)^T \quad \text{4D vector of canonical variables}$$

$$\theta_x = x' - Ry / 2$$

$$\theta_y = y' + Rx / 2$$

$$R = eB_s / Pc$$

Hamiltonian

- For a flat horizontal orbit, the Hamiltonian matrix is

$$\mathbf{H} = \begin{pmatrix} K^2 + k + \frac{R^2}{4} & 0 & k_s & -R/2 \\ 0 & 1 & R/2 & 0 \\ k_s & R/2 & -k + \frac{R^2}{4} & 0 \\ -R/2 & 0 & 0 & 1 \end{pmatrix}$$

$$K = eB_y / Pc$$

Dipole

$$k = eG / Pc$$

Quad

$$k_s = eG_s / Pc$$

Skew quad

For any 2 solutions $\mathbf{x}_1^T \mathbf{U} \mathbf{x}_2 = \text{const}$ - Lagrange invariants

Symplecticity

- To preserve Lagrange invariants, the revolution matrix **M** must be symplectic:

$$\mathbf{M}^T \mathbf{U} \mathbf{M} = \mathbf{U} \iff \mathbf{M} \mathbf{U} \mathbf{M}^T = \mathbf{U}$$

- This leaves only 10 independent parameters for 16 matrix elements.
- For the eigenvectors and eigenvalues

$$\mathbf{M} \mathbf{v}_i = \lambda_i \mathbf{v}_i \quad , i = 1, \dots, 4$$

- $|\mathbf{M}| = 1 \implies \lambda_1 \lambda_2 \lambda_3 \lambda_4 = 1$ Stability: $|\lambda_i| = 1$

- **M** is real: $\lambda_3 = \lambda_1^*$; $\lambda_4 = \lambda_2^*$ $\implies \lambda_{1,2} = \exp(-i\mu_{1,2})$

Orthogonality and normalization

- From the symplecticity, for any 2 eigenvectors, symplectic orthogonality follows:

$$\mathbf{v}_k^\dagger \mathbf{U} \mathbf{v}_m = 0, \quad k \neq m = 1, \dots, 4$$

- With normalization $\mathbf{v}_l^\dagger \mathbf{U} \mathbf{v}_l = -2i, \quad l = 1, 2$ the eigen-vector matrix

$$\mathbf{V} = [\text{Re } \mathbf{v}_1, -\text{Im } \mathbf{v}_1, \text{Re } \mathbf{v}_2, -\text{Im } \mathbf{v}_2]$$

is symplectic.

General Solution

- Turn-by-turn particle positions and angles:

$$\mathbf{x} = \text{Re} \left(\sqrt{2J_1} e^{-i\psi_1} \mathbf{v}_1 + \sqrt{2J_2} e^{-i\psi_2} \mathbf{v}_2 \right) = \mathbf{V}\boldsymbol{\xi}$$

$$\boldsymbol{\xi} = \begin{pmatrix} \sqrt{2J_1} \cos \psi_1 \\ -\sqrt{2J_1} \sin \psi_1 \\ \sqrt{2J_2} \cos \psi_2 \\ -\sqrt{2J_2} \sin \psi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \pi_1 \\ \xi_2 \\ \pi_2 \end{pmatrix}$$

- Transformation $\mathbf{x} \rightarrow \boldsymbol{\xi}$ is canonical,

$$H = \frac{\mu_1}{C} \left(\frac{\xi_1^2}{2} + \frac{\pi_1^2}{2} \right) + \frac{\mu_2}{C} \left(\frac{\xi_2^2}{2} + \frac{\pi_2^2}{2} \right) = \frac{\mu_1}{C} J_1 + \frac{\mu_2}{C} J_2$$

- Hamiltonian

$J_{1,2}$ - actions

$\psi_{1,2}$ - phases

$\mu_{1,2} = 2\pi\nu_{1,2}$ - phase advances

Emittances

- RMS emittances: $\langle J_{1,2} \rangle \equiv \varepsilon_{1,2}$

- Matched 4D ellipsoid: $\mathbf{x}^T \mathbf{\Xi} \mathbf{x} = 1$

$$\mathbf{\Xi} = -\mathbf{U} \mathbf{V} \mathbf{E}^{-1} \mathbf{V}^T \mathbf{U}$$

$$\mathbf{E} = \text{Diag}(\varepsilon_1, \varepsilon_1, \varepsilon_2, \varepsilon_2)$$

- The quadratic form $\mathbf{\Xi}$ determines the emittances and eigenvectors:

$$(\mathbf{\Xi} - i\varepsilon_l^{-1} \mathbf{U}) \mathbf{v}_l = 0; \quad \det(\mathbf{\Xi} - i\varepsilon_l^{-1} \mathbf{U}) = 0.$$

Sigma Matrix

- For a Gaussian distribution

$$f(\mathbf{x}) = (4\pi^2 \varepsilon_1 \varepsilon_2)^{-1} \exp(-\mathbf{x}^T \mathbf{\Xi} \mathbf{x} / 2)$$

the second-order moments are:

$$\Sigma_{ij} \equiv \langle x_i x_j \rangle = \int x_i x_j f(\mathbf{x}) dx^4 = (\mathbf{V} \mathbf{E} \mathbf{V}^T)_{ij} = (\mathbf{\Xi}^{-1})_{ij}$$

- The emittances and the eigenvectors can be also found from the Σ matrix:

$$\det(\Sigma \mathbf{U} + i \varepsilon_l \mathbf{I}) = 0; \quad (\Sigma \mathbf{U} + i \varepsilon_l \mathbf{I}) \hat{\mathbf{v}}_l = 0.$$

- The mode emittances ε_1 and ε_2 are the motion invariants, i.e. they cannot be changed in the course of linear Hamiltonian motion.

Eigenvectors

- Mais-Ripken parameterization:

$$\mathbf{v}_1 = \left(\sqrt{\beta_{1x}}, -\frac{i(1-u) + \alpha_{1x}}{\sqrt{\beta_{1x}}}, \sqrt{\beta_{1y}} e^{iv_1}, -\frac{i u + \alpha_{1y}}{\sqrt{\beta_{1y}}} e^{iv_1} \right)^T,$$
$$\mathbf{v}_2 = \left(\sqrt{\beta_{2x}} e^{iv_2}, -\frac{i u + \alpha_{2x}}{\sqrt{\beta_{2x}}} e^{iv_2}, \sqrt{\beta_{2y}}, -\frac{i(1-u) + \alpha_{2y}}{\sqrt{\beta_{2y}}} \right)^T.$$

- Transfer matrix $s_i \rightarrow s_f$: $\mathbf{M}(s_i, s_f) = -\mathbf{V}(s_f) \mathbf{S} \mathbf{U} \mathbf{V}(s_i)^T \mathbf{U} = \mathbf{V}(s_f) \mathbf{S} \mathbf{V}(s_i)^{-1}$

$$\mathbf{S} = \begin{pmatrix} \cos \Delta \psi_1 & \sin \Delta \psi_1 & 0 & 0 \\ -\sin \Delta \psi_1 & \cos \Delta \psi_1 & 0 & 0 \\ 0 & 0 & \cos \Delta \psi_2 & \sin \Delta \psi_2 \\ 0 & 0 & -\sin \Delta \psi_2 & \cos \Delta \psi_2 \end{pmatrix}$$

- Any matrix constructed in that way from 2 sets of eigenvectors is symplectic, i.e. doable.

Circular modes

- With $\beta_{lx}=\beta_{ly}=\beta$, $\alpha_{lx}=\alpha_{ly}=\alpha$, $u=1/2$, and $v_{1,2}=\pi/2$:

$$\mathbf{v}_1 = \left(\sqrt{\beta}, -\frac{i/2 + \alpha}{\sqrt{\beta}}, i\sqrt{\beta}, -i\frac{i/2 + \alpha}{\sqrt{\beta}} \right)^T,$$
$$\mathbf{v}_2 = \left(i\sqrt{\beta}, -i\frac{i/2 + \alpha}{\sqrt{\beta}}, \sqrt{\beta}, -\frac{i/2 + \alpha}{\sqrt{\beta}} \right)^T.$$

- In a matched solenoid one of modes is a Larmor motion with center at the solenoid axis, and another one is a pure offset, $x, y = \text{const}$.

$$\boldsymbol{\varepsilon}_1 \boldsymbol{\varepsilon}_2 = \boldsymbol{\varepsilon}_{4D}; \quad \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 = \langle x\boldsymbol{\theta}_y - y\boldsymbol{\theta}_x \rangle$$

Beam Adapters

- Circular and planar modes can be transferred one into another (Derbenev).
- Example: a transfer matrix

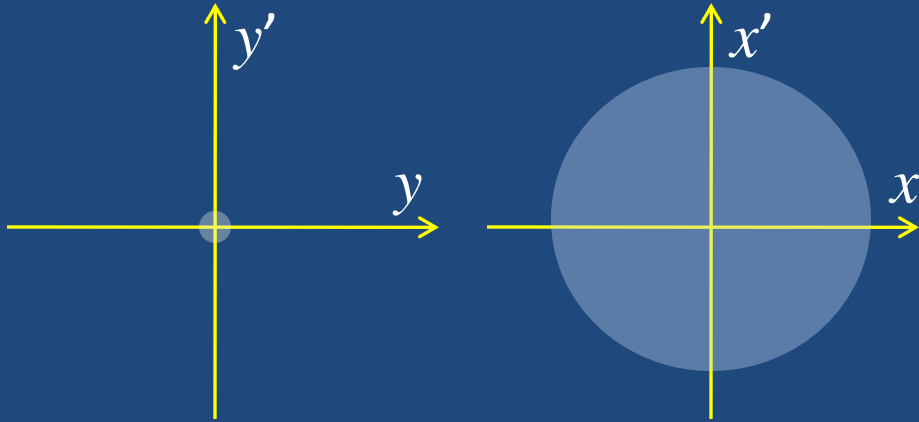
$$\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{bmatrix}; \quad \mathbf{N} = \begin{pmatrix} 0 & 2\beta \\ -(2\beta)^{-1} & 0 \end{pmatrix}; \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

transforms 2 circular modes with $\alpha = 0$ into 2 planar modes tilted by 45° . Thus, the same transfer line tilted by 45° yields normal x-y planar modes.

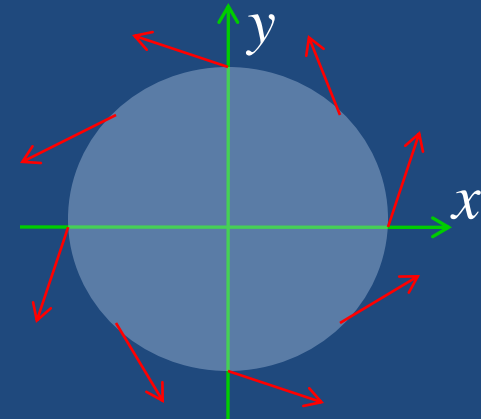
- Adapter implementation requires 3 skew quads.

Planar-Circular mode transformation

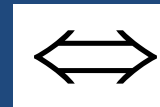
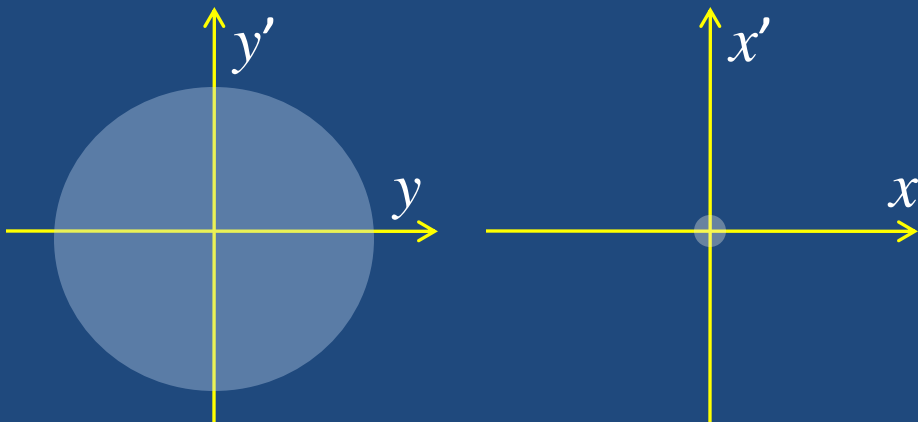
X-mode



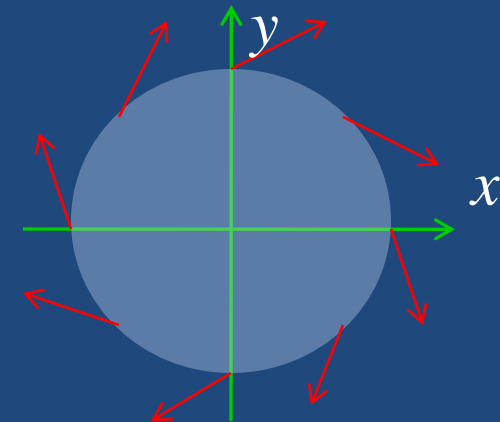
Counter-clockwise



Y-mode



Clockwise



Perturbation theory

- Let the revolution matrix \mathbf{M} be perturbed: $\mathbf{M} = (\mathbf{I} + \mathbf{P})\mathbf{M}_0$

where the perturbation \mathbf{P} is small, but **not necessarily symplectic**.

- In the first order of perturbation theory, the complex tune shifts are

$$\Delta\mu_l / (2\pi) = -(4\pi)^{-1} \mathbf{v}_l^\dagger \mathbf{U} \mathbf{P} \mathbf{v}_l, \quad l = 1, 2.$$

Similar to Quantum Mechanics!

- In particular, this formula allows calculation of the incoherent beam-beam and space charge tune shifts for arbitrary-coupled optics.
- The rate-sum theorem:** sum of the two growth rates is independent on the eigenvectors:

$$\text{Im}(\Delta\mu_1 + \Delta\mu_2) = \text{Tr}(\mathbf{P}) / 2.$$

Coherent motion

- If the coherent tune shifts are small enough,

$$|\Delta Q_c| \ll |Q_1 - Q_2|$$

coupling is taken into account by the following rule of correspondence

$$\beta_{x,y} Z_{x,y} \rightarrow \beta_{lx} Z_x + \beta_{ly} Z_y ; l = 1, 2$$

$$Q_{x,y} \rightarrow Q_{1,2}$$

- After that, all the formulas of the uncoupled theory are applicable.

Space charge suppression

- For a conventional uncoupled planar modes, the SC tune shifts:

$$\Delta Q_{1,2} = -\frac{\lambda r_0}{2\pi\gamma_0^3\beta_0^2} \oint \frac{\beta_{x,y} ds}{a_{1,2}(a_1 + a_2)} ; \quad a_{1,2} = \sqrt{\varepsilon_{1,2}\beta_{1,2}}$$

- For $\varepsilon_1 \gg \varepsilon_2$, smooth approximation and equal betas:

$$\Delta Q_2|_{\text{planar}} = -\frac{\lambda r_0 C}{2\pi\gamma_0^3\beta_0^2\sqrt{\varepsilon_1\varepsilon_2}} = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} \Delta Q_1|_{\text{planar}} .$$

- The same approximation for the circular optics yields

$$\Delta Q_2|_{\text{circular}} = \Delta Q_1|_{\text{circular}} = \frac{\lambda r_0 C}{2\pi\gamma_0^3\beta_0^2\varepsilon_1}$$

- For the circular optics, the tune shifts are finite even for $\varepsilon_2 = 0$!

$$\frac{\Delta Q|_{\text{circular}}}{\Delta Q_2|_{\text{planar}}} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \ll 1$$

- Can it be used for LHC upgrades?

Circular optics for IP

- When IP optics is circular, angular momentum is preserved in the beam-beam interactions, thus higher beam-beam parameters are available.
- For VEPP-2000 e+e- ring (BINP, Novosibirsk) with circular optics at 2 IPs, the beam-beam parameter reaches as high as

$$\Delta Q_{\text{bb}} = 0.08$$

per each of them (strong-strong regime)!

- Can it be used for LHC upgrades?

Flat beams at LHC?

- Back to flat beams, discussed for the space charge suppression:

$$\varepsilon_1 \gg \varepsilon_2$$

- How we can better use them in the LHC?
- One option is redistribution of the emittances by means of the Montague resonance, and use circular IP with

$$\varepsilon_1, \varepsilon_2 \rightarrow (\varepsilon_1 + \varepsilon_2) / 2, (\varepsilon_1 + \varepsilon_2) / 2 \quad .$$

- Another option – flat beams at the IP, with the luminosity

$$\mathcal{L} \propto \frac{1}{\sqrt{\varepsilon_2}}$$

Many thanks for everyone of you!