Beam Physics with Coupled Optics

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Equations of Motion

- For a linear Hamiltonian system with two degrees of freedom, equations of motion can be written in the following matrix form:

$$\frac{d\mathbf{x}}{ds} = \mathbf{UHx}\,, \quad \mathbf{U} = \begin{bmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \end{bmatrix}, \quad \mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\mathbf{H}(s)_{ik} = \partial^2 H / \partial x_i \partial x_k$$ Hessian matrix of the Hamiltonian

$$\mathbf{x} = (x, \theta_x, y, \theta_y)^T$$ 4D vector of canonical variables

$$\theta_x = x' - Ry/2 \qquad \theta_y = y' + Rx/2 \qquad R = eB_s / Pc$$

Hamiltonian

- For a flat horizontal orbit, the Hamiltonian matrix is

$$\mathbf{H} = \begin{pmatrix} K^2 + k + \frac{R^2}{4} & 0 & k_s & -R/2 \\ 0 & 1 & R/2 & 0 \\ k_s & R/2 & -k + \frac{R^2}{4} & 0 \\ -R/2 & 0 & 0 & 1 \end{pmatrix}$$

$K = eB_y / Pc$ Dipole

$k = eG / Pc$ Quad

$k_s = eG_s / Pc$ Skew quad

For any 2 solutions $\mathbf{x}_1{}^T \mathbf{U} \mathbf{x}_2 = \text{const}$ - Lagrange invariants

Symplecticity

- To preserve Lagrange invariants, the revolution matrix $\mathbf{M}$ must be symplectic:

$$\mathbf{M}^T\mathbf{U}\mathbf{M} = \mathbf{U} \quad \Leftrightarrow \quad \mathbf{M}\mathbf{U}\mathbf{M}^T = \mathbf{U}$$

- This leaves only 10 independent parameters for 16 matrix elements.

- For the eigenvectors and eigenvalues

$$\mathbf{M}\mathbf{v}_i = \lambda_i \mathbf{v}_i \quad , i = 1,\ldots,4$$

- $|\,\mathbf{M}\,| = 1 \quad \Rightarrow \quad \lambda_1\lambda_2\lambda_3\lambda_4 = 1$ Stability: $|\,\lambda_i\,| = 1$

- $\mathbf{M}$ is real: $\lambda_3 = \lambda_1^*;\ \lambda_4 = \lambda_2^* \quad \Rightarrow \quad \lambda_{1,2} = \exp(-i\mu_{1,2})$

Orthogonality and normalization

- From the symplecticity, for any 2 eigenvectors, symplectic orthogonality follows:

$$\mathbf{v}_k{}^{\dagger}\mathbf{U}\mathbf{v}_m = 0, \quad k \neq m = 1,\ldots,4$$

- With normalization $\mathbf{v}_l{}^{\dagger}\mathbf{U}\mathbf{v}_l = -2i, \quad l = 1,2$ the eigen-vector matrix

$$\mathbf{V} = \left[\mathrm{Re}\,\mathbf{v}_1, -\mathrm{Im}\,\mathbf{v}_1, \mathrm{Re}\,\mathbf{v}_2, -\mathrm{Im}\,\mathbf{v}_2\right]$$

is symplectic.

General Solution

- Turn-by-turn particle positions and angles:

$$\mathbf{x} = \mathrm{Re}\left(\sqrt{2J_1}e^{-i\psi_1}\mathbf{v}_1 + \sqrt{2J_2}e^{-i\psi_2}\mathbf{v}_2\right) = \mathbf{V}\cdot\boldsymbol{\xi}$$

$$\boldsymbol{\xi} = \begin{pmatrix} \sqrt{2J_1}\cos\psi_1 \\ -\sqrt{2J_1}\sin\psi_1 \\ \sqrt{2J_2}\cos\psi_2 \\ -\sqrt{2J_2}\sin\psi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \pi_1 \\ \xi_2 \\ \pi_2 \end{pmatrix}$$

- Transformation $\mathbf{x} \rightarrow \boldsymbol{\xi}$ is canonical,

$$H = \frac{\mu_1}{C}\left(\frac{\xi_1^2}{2} + \frac{\pi_1^2}{2}\right) + \frac{\mu_2}{C}\left(\frac{\xi_2^2}{2} + \frac{\pi_2^2}{2}\right) = \frac{\mu_1}{C}J_1 + \frac{\mu_2}{C}J_2$$

- Hamiltonian

$J_{1,2}$ - actions

$\psi_{1,2}$ - phases

$\mu_{1,2} = 2\pi\nu_{1,2}$ - phase advances

Emittances

- RMS emittances: $\left\langle J_{1,2} \right\rangle \equiv \varepsilon_{1,2}$
- Matched 4D ellipsoid: $\mathbf{x}^T \boldsymbol{\Xi} \mathbf{x} = 1$

$$\boldsymbol{\Xi} = -\mathbf{UVE}^{-1}\mathbf{V}^T\mathbf{U}$$

$$\mathbf{E} = \mathrm{Diag}(\varepsilon_1, \varepsilon_1, \varepsilon_2, \varepsilon_2)$$

- The quadratic form $\boldsymbol{\Xi}$ determines the emittances and eigenvectors:

$$\left(\boldsymbol{\Xi} - i\varepsilon_l^{-1}\mathbf{U}\right)\mathbf{v}_l = 0\,; \quad \det\left(\boldsymbol{\Xi} - i\varepsilon_l^{-1}\,\mathbf{U}\right) = 0\,.$$

Sigma Matrix

- For a Gaussian distribution

$$f(\mathbf{x}) = \left(4\pi^2\varepsilon_1\varepsilon_2\right)^{-1}\exp\left(-\mathbf{x}^T\boldsymbol{\Xi}\mathbf{x}/2\right)$$

the second-order moments are:

$$\boldsymbol{\Sigma}_{ij} \equiv \left\langle x_i x_j \right\rangle = \int x_i x_j f(\mathbf{x}) dx^4 = \left(\mathbf{V}\mathbf{E}\mathbf{V}^T\right)_{ij} = \left(\boldsymbol{\Xi}^{-1}\right)_{ij}$$

- The emittances and the eigenvectors can be also found from the Σ matrix:

$$\det\left(\boldsymbol{\Sigma}\mathbf{U} + i\varepsilon_l\,\mathbf{I}\right) = 0\,; \quad \left(\boldsymbol{\Sigma}\mathbf{U} + i\varepsilon_l\mathbf{I}\right)\hat{\mathbf{v}}_l = 0.$$

- The mode emittances ε_1 and ε_2 are the motion invariants, i.e. they cannot be changed in the course of linear Hamiltonian motion.

Eigenvectors

- Mais-Ripken parameterization:

$$\mathbf{v}_1 = \left(\sqrt{\beta_{1x}}, -\frac{i(1-u)+\alpha_{1x}}{\sqrt{\beta_{1x}}}, \sqrt{\beta_{1y}} e^{i\nu_1}, -\frac{iu+\alpha_{1y}}{\sqrt{\beta_{1y}}} e^{i\nu_1} \right)^T,$$

$$\mathbf{v}_2 = \left(\sqrt{\beta_{2x}} e^{i\nu_2}, -\frac{iu+\alpha_{2x}}{\sqrt{\beta_{2x}}} e^{i\nu_2}, \sqrt{\beta_{2y}}, -\frac{i(1-u)+\alpha_{2y}}{\sqrt{\beta_{2y}}} \right)^T.$$

- Transfer matrix $s_\mathrm{i} \to s_\mathrm{f}$: $\mathbf{M}(s_\mathrm{i}, s_\mathrm{f}) = -\mathbf{V}(s_\mathrm{f})\mathbf{SUV}(s_\mathrm{i})^\mathbf{T}\mathbf{U} = \mathbf{V}(s_\mathrm{f})\mathbf{SV}(s_\mathrm{i})^{-1}$

$$\mathbf{S} = \begin{pmatrix} \cos\Delta\psi_1 & \sin\Delta\psi_1 & 0 & 0 \\ -\sin\Delta\psi_1 & \cos\Delta\psi_1 & 0 & 0 \\ 0 & 0 & \cos\Delta\psi_2 & \sin\Delta\psi_2 \\ 0 & 0 & -\sin\Delta\psi_2 & \cos\Delta\psi_2 \end{pmatrix}$$

- Any matrix constructed in that way from 2 sets of eigenvectors is symplectic, i.e. doable.

Circular modes

- With $\beta_{lx}=\beta_{ly}=\beta, \quad \alpha_{lx}=\alpha_{ly}=\alpha, \quad u=1/2,$ and $\nu_{1,2}=\pi/2$:

$$\mathbf{v}_1 = \left(\sqrt{\beta}, -\frac{i/2+\alpha}{\sqrt{\beta}}, i\sqrt{\beta}, -i\frac{i/2+\alpha}{\sqrt{\beta}} \right)^T ,$$

$$\mathbf{v}_2 = \left(i\sqrt{\beta}, -i\frac{i/2+\alpha}{\sqrt{\beta}}, \sqrt{\beta}, -\frac{i/2+\alpha}{\sqrt{\beta}} \right)^T .$$

- In a matched solenoid one of modes is a Larmor motion with center at the solenoid axis, and another one is a pure offset, $x, y = const$.

$$\varepsilon_1\varepsilon_2 = \varepsilon_{4D}; \quad \varepsilon_1 - \varepsilon_2 = \left\langle x\theta_y - y\theta_x \right\rangle$$

Beam Adapters

- Circular and planar modes can be transferred one into another (Derbenev).

- Example: a transfer matrix

$$\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{bmatrix}; \quad \mathbf{N} = \begin{pmatrix} 0 & 2\beta \\ -(2\beta)^{-1} & 0 \end{pmatrix}; \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

  transforms 2 circular modes with $\alpha = 0$ into 2 planar modes tilted by 45°. Thus, the same transfer line tilted by 45° yields normal x-y planar modes.

- Adapter implementation requires 3 skew quads.

Perturbation theory

- Let the revolution matrix $\mathbf{M}$ be perturbed: $\mathbf{M} = (\mathbf{I} + \mathbf{P})\mathbf{M}_0$

  where the perturbation $\mathbf{P}$ is small, but not necessarily symplectic.

- In the first order of perturbation theory, the complex tune shifts are

$$\Delta\mu_l / (2\pi) = -(4\pi)^{-1} \; \mathbf{v}_l^{\dagger} \mathbf{U} \, \mathbf{P} \, \mathbf{v}_l \; , \qquad l = 1, 2 \, .$$

- In particular, this formula allows calculation of the incoherent beam-beam and space charge tune shifts for arbitrary-coupled optics.

- The rate-sum theorem: sum of the two growth rates is independent on the eigenvectors:

$$\mathrm{Im}(\Delta\mu_1 + \Delta\mu_2) = \mathrm{Tr}(\mathbf{P}) / 2 \, .$$

Coherent motion

- If the coherent tune shifts are small enough,

$$|\Delta Q_c| \ll |Q_1 - Q_2|$$

coupling is taken into account by the following rule of correspondence

$$\beta_{x,y} Z_{x,y} \rightarrow \beta_{lx} Z_x + \beta_{ly} Z_y \quad ; l = 1,2$$

$$Q_{x,y} \rightarrow Q_{1,2}$$

- After that, all the formulas of the uncoupled theory are applicable.

Space charge suppression

- For a conventional uncoupled planar modes, the SC tune shifts:

$$\Delta Q_{1,2} = -\frac{\lambda r_0}{2\pi\gamma_0^3\beta_0^2}\oint \frac{\beta_{x,y}ds}{a_{1,2}(a_1+a_2)}\ ; \quad a_{1,2} = \sqrt{\varepsilon_{1,2}\beta_{1,2}}$$

- For $\varepsilon_1 \gg \varepsilon_2$, smooth approximation and equal betas:

$$\Delta Q_2\big|_{\text{planar}} = -\frac{\lambda r_0 C}{2\pi\gamma_0^3\beta_0^2\sqrt{\varepsilon_1\varepsilon_2}} = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\,\Delta Q_1\big|_{\text{planar}}\ .$$

- The same approximation for the circular optics yields

$$\Delta Q_2\big|_{\text{circular}} = \Delta Q_1\big|_{\text{circular}} = \frac{\lambda r_0 C}{2\pi\gamma_0^3\beta_0^2\varepsilon_1}$$

- For the circular optics, the tune shifts are finite even for $\varepsilon_2 = 0$!

$$\frac{\Delta Q\big|_{\text{circular}}}{\Delta Q_2\big|_{\text{planar}}} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \ll 1$$

- Can it be used for LHC upgrades?

Circular optics for IP

- When IP optics is circular, angular momentum is preserved in the beam-beam interactions, thus higher beam-beam parameters are available.

- For VEPP-2000 e+e- ring (BINP, Novosibirsk) with circular optics at 2 IPs, the beam-beam parameter reaches as high as

$$\Delta Q_{\mathrm{bb}} = 0.08$$

per each of them (strong-strong regime)!

- Can it be used for LHC upgrades?

Many thanks for everyone of you!