Part III

The Synchro-Betatron Hamiltonian

Outline

5 The Synchro-Betatron Hamiltonian
- Series of canonical transformations
- RF fields
- The full Synchro-Betatron Hamiltonian

Canonical transformation to kinetic energy (1)

Hamiltonian

$$
\begin{aligned}
& H(u, P_u, s_0, H_s; s) = \\
& - \left(1 + \frac{x}{R}\right) \sqrt{\frac{(E_0\beta_0^2 H_s - qV)^2/c^2 - m^2c^2}{P_0^2} - P_u^2} \\
& - \frac{1}{B_0 R} \left(1 + \frac{x}{R}\right) A_s
\end{aligned}
\tag{9}
$$

$P^2 = \frac{(H_0 - qV)^2}{c^2} - m^2c^2$, $P_0 = \frac{E_0\beta_0}{c}$

Canonical transformation to kinetic energy (1)

Hamiltonian

$$
\begin{aligned}
H(u, P_u, s_0, H_s; s) = \\
&- \left(1 + \frac{x}{R}\right) \sqrt{\frac{(E_0\beta_0^2 H_s - qV)^2/c^2 - m^2c^2}{P_0^2} - P_u^2} \\
&- \frac{1}{B_0 R} \left(1 + \frac{x}{R}\right) A_s
\end{aligned}
\tag{9}
$$

$P^2 = \frac{(H_0 - qV)^2}{c^2} - m^2c^2$, $P_0 = \frac{E_0\beta_0}{c}$

Kinetic energy

$$
E_0\beta_0^2 E = E_0\beta_0^2 H_s - qV \tag{10}
$$

Canonical transformation to kinetic energy (2)

Generating function

$$
\begin{aligned}
F_2(u, \tilde{P}_u, s_0, E; s) &= u\tilde{P}_u + \int H_s(s_0, E)\, ds_0 \\
&= u\tilde{P}_u + E s_0 + \frac{q}{E_0\beta_0^2} \int V\, ds_0
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial F_2}{\partial u} = P_u = \tilde{P}_u - \frac{q}{E_0\beta_0^2} \int E_u\, ds_0\,, \quad \frac{\partial F_2}{\partial \tilde{P}_u} = \tilde{u} = u \\
&\frac{\partial F_2}{\partial s_0} = H_s\,, \quad \frac{\partial F_2}{\partial E} = s_E = s_0 \\
&\frac{\partial F_2}{\partial s} = -H_K = -\frac{q}{E_0\beta_0^2} \left(1 + \frac{x}{R}\right) \int E_s\, ds_0
\end{aligned}
$$

$E_u = -\frac{\partial}{\partial u} V$, $E_s = -\frac{1}{1+x/R} \frac{\partial}{\partial s} V$

Canonical transformation to kinetic energy (3)

Hamiltonian

$$\tilde{H}(u, \tilde{P}_u, s_E, E; s) = -\left(1 + \frac{x}{R}\right) \times \sqrt{\frac{E_0^2 \beta_0^4 E^2/c^2 - m^2 c^2}{P_0^2} - \left(\tilde{P}_u - \frac{q}{E_0 \beta_0^2} \int E_u \, ds_0\right)^2} - \frac{1}{B_0 R}\left(1 + \frac{x}{R}\right)\left(A_s - \bar{E}_s\right) \tag{11}$$

$\bar{E}_s = \frac{1}{\beta_0 c} \int E_s \, ds_0$

Canonical transformation to kinetic energy (3)

Hamiltonian

$$
\begin{aligned}
\tilde{H}(u, \tilde{P}_u, s_E, E; s) = &-\left(1 + \frac{x}{R}\right) \\
&\times \sqrt{\frac{E_0^2 \beta_0^4 E^2/c^2 - m^2c^2}{P_0^2} - \left(\tilde{P}_u - \frac{q}{E_0\beta_0^2}\int E_u \, ds_0\right)^2} \\
&- \frac{1}{B_0 R}\left(1 + \frac{x}{R}\right)\left(A_s - \bar{E}_s\right)
\end{aligned}
\tag{11}
$$

$\bar{E}_s = \frac{1}{\beta_0 c}\int E_s \, ds_0$

The transverse fields are usually small and vanish upon averaging

$$
\tilde{P}_u - \frac{q}{E_0\beta_0^2}\int E_u \, ds_0 \approx \tilde{P}_u
$$

Canonical transformation to kinetic energy (4)

We immediately relabel all "∼"-variables and can write

Hamiltonian

$$H(u, P_u, s_E, E; s) = -\left(1 + \frac{x}{R}\right)\sqrt{\beta_0^2 E^2 - \frac{1}{\gamma_0^2 \beta_0^2} - P_u^2} - \frac{1}{B_0 R}\left(1 + \frac{x}{R}\right)\left(A_s - \bar{E}_s\right) \tag{12}$$

$P_0 = \frac{E_0 \beta_0}{c}$

Canonical transformation to reference orbit system (1)

Hamiltonian

$$H(u, P_u, s_E, E; s) = -\left(1 + \frac{x}{R}\right)\sqrt{\beta_0^2 E^2 - \frac{1}{\gamma_0^2 \beta_0^2} - P_u^2} - \frac{1}{B_0 R}\left(1 + \frac{x}{R}\right)\left(A_s - \bar{E}_s\right) \tag{13}$$

$P_0 = \frac{E_0 \beta_0}{c}$

Reference orbit

$$\sigma = s - s_E$$

Canonical transformation to reference orbit system (2)

Generating function

$$\begin{aligned} F_3(\tilde{u}, P_u, \sigma, E; s) &= -\tilde{u}P_u - \int s_E(\sigma, E)\, dE \\ &= -\tilde{u}P_u + \sigma E - \sigma \frac{1}{\beta_0^2} - sE + s\frac{1}{\beta_0^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial F_3}{\partial \tilde{u}} &= -\tilde{P}_u = -P_u \,, \quad \frac{\partial F_3}{\partial P_u} = -u = -\tilde{u} \\ \frac{\partial F_3}{\partial \sigma} &= -\delta = -\left(E - \frac{1}{\beta_0^2}\right) = -\frac{1}{\beta_0^2}\left(\frac{\gamma}{\gamma_0} - 1\right) \,, \quad \frac{\partial F_3}{\partial E} = -s_E \\ \frac{\partial F_3}{\partial s} &= -\delta = -\left(E - \frac{1}{\beta_0^2}\right) \end{aligned}$$

$E = \frac{H_0 - qV}{E_0 \beta_0^2}$

Canonical transformation to reference orbit system (2)

Generating function

$$F_3(\tilde{u}, P_u, \sigma, E; s) = -\tilde{u}P_u - \int s_E(\sigma, E)\, dE$$
$$= -\tilde{u}P_u + \sigma E - \sigma \frac{1}{\beta_0^2} - sE + s\frac{1}{\beta_0^2}$$

$$\frac{\partial F_3}{\partial \tilde{u}} = -\tilde{P}_u = -P_u\,, \quad \frac{\partial F_3}{\partial P_u} = -u = -\tilde{u}$$
$$\frac{\partial F_3}{\partial \sigma} = -\delta = -\left(E - \frac{1}{\beta_0^2}\right) = -\frac{1}{\beta_0^2}\left(\frac{\gamma}{\gamma_0} - 1\right)\,, \quad \frac{\partial F_3}{\partial E} = -s_E$$
$$\frac{\partial F_3}{\partial s} = -\delta = -\left(E - \frac{1}{\beta_0^2}\right)$$

$E = \frac{H_0 - qV}{E_0 \beta_0^2}$

Canonical transformation to reference orbit system (3)

We immediately relabel all "$\sim$"-variables

Hamiltonian

$$H(u, P_u, \sigma, \delta; s) = -\left(1 + \frac{x}{R}\right) \sqrt{\beta_0^2 \left(\delta + \frac{1}{\beta_0^2}\right)^2 - \frac{1}{\gamma_0^2 \beta_0^2} - P_u^2} - \frac{1}{B_0 R} \left(1 + \frac{x}{R}\right) (A_s - \bar{E}_s) + \delta \tag{14}$$

Expansions around reference orbit system (1)

We set $\Gamma(\delta)^2 = \beta_0^2 \left(\delta + \frac{1}{\beta_0^2}\right)^2 - \frac{1}{\gamma_0^2 \beta_0^2}$ and obtain

Hamiltonian

$$H = -\left(1 + \frac{x}{R}\right) \sqrt{\Gamma(\delta)^2 - P_u^2} - \frac{1}{B_0 R} \left(1 + \frac{x}{R}\right) \left(A_s - \bar{E}_s\right) + \delta \tag{15}$$

Expansions around reference orbit system (2)

- Taylor expansion in P_u:

$$\sqrt{\Gamma^2 - P_u^2} \approx \Gamma - \frac{1}{2\Gamma} P_u^2$$

- Taylor expansion in δ:

$$\sqrt{\beta_0^2 \left(\delta + \frac{1}{\beta_0^2}\right)^2 - \frac{1}{\gamma_0^2 \beta_0^2}} \approx 1 + \delta - \frac{\delta^2}{2\gamma_0^2} + \frac{\delta^3}{2\gamma_0^2}$$

- Taylor expansion in δ:

$$\frac{1}{\sqrt{\beta_0^2 \left(\delta + \frac{1}{\beta_0^2}\right)^2 - \frac{1}{\gamma_0^2 \beta_0^2}}} \approx 1 - \delta + \left(1 + \frac{1}{2\gamma_0^2}\right)\delta^2$$

Canonical transformation to closed orbit system (1)

Hamiltonian

$$
\begin{aligned}
H &\approx -\left(1+\frac{x}{R}\right)\left(\Gamma-\frac{1}{2\Gamma}P_u^2\right)-\frac{1}{B_0R}\left(1+\frac{x}{R}\right)\left(A_s-\bar{E}_s\right)+\delta \\
&= -\Gamma-\frac{x}{R}\Gamma+\frac{1}{2\Gamma}P_u^2+\frac{1}{2\Gamma}\frac{x}{R}P_u^2 \\
&\quad -\frac{1}{B_0R}\left(1+\frac{x}{R}\right)\left(A_s-\bar{E}_s\right)+\delta
\end{aligned}
$$

Identification of terms

Hamiltonian

$$H = -\Gamma - \frac{x}{R}\Gamma + \frac{1}{2\Gamma}P_u^2 + \frac{1}{2\Gamma}\frac{x}{R}P_u^2 - \frac{1}{B_0 R}\left(1 + \frac{x}{R}\right)\left(A_s - \bar{E}_s\right) + \delta$$

- Dispersion: linear synchro-betatron coupling
- Chromaticity: terms $\sim P_u^2\,\delta^k$
- Magnetic field terms

$\rightarrow$ Eliminate the first order synchro-betatron coupling by moving to the closed orbit system $(\hat{x}, \hat{p}_x, \hat{\sigma}, \hat{\delta})$ where all first order synchro-betatron terms $\sim \hat{x}\delta^k$ and $\sim \hat{p}_x\delta^k$ cancel; this will naturally introduce dispersion

Identification of terms

Hamiltonian

$$H = -\Gamma - \frac{x}{R}\Gamma + \frac{1}{2\Gamma}P_u^2 + \frac{1}{2\Gamma}\frac{x}{R}P_u^2$$
$$- \frac{1}{B_0 R}\left(1 + \frac{x}{R}\right)(A_s - \bar{E}_s) + \delta$$

- Dispersion: linear synchro-betatron coupling
- Chromaticity: terms $\sim P_u^2\,\delta^k$
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$\rightarrow$ Eliminate the first order synchro-betatron coupling by moving to the closed orbit system $(\hat{x}, \hat{p}_x, \hat{\sigma}, \hat{\delta})$ where all first order synchro-betatron terms $\sim \hat{x}\delta^k$ and $\sim \hat{p}_x\delta^k$ cancel; this will naturally introduce dispersion

Identification of terms

Hamiltonian

$$H = -\Gamma - \frac{x}{R}\Gamma + \frac{1}{2\Gamma}P_u^2 + \frac{1}{2\Gamma}\frac{x}{R}P_u^2 - \frac{1}{B_0 R}\left(1 + \frac{x}{R}\right)\left(A_s - \bar{E}_s\right) + \delta$$

- Dispersion: linear synchro-betatron coupling
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Identification of terms

Hamiltonian

$$H = -\Gamma - \frac{x}{R}\Gamma + \frac{1}{2\Gamma}P_u^2 + \frac{1}{2\Gamma}\frac{x}{R}P_u^2 - \frac{1}{B_0 R}\left(1 + \frac{x}{R}\right)(A_s - \bar{E}_s) + \delta$$

- Dispersion: linear synchro-betatron coupling
- Chromaticity: terms $\sim P_u^2\,\delta^k$
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$\rightarrow$ Eliminate the first order synchro-betatron coupling by moving to the closed orbit system $(\hat{x}, \hat{p}_x, \hat{\sigma}, \hat{\delta})$ where all first order synchro-betatron terms $\sim \hat{x}\delta^k$ and $\sim \hat{p}_x\delta^k$ cancel; this will naturally introduce dispersion

Identification of terms

Hamiltonian

$$H = -\Gamma - \frac{x}{R}\Gamma + \frac{1}{2\Gamma}P_u^2 + \delta$$
$$+ \frac{1}{2}K^2 x^2 + \frac{1}{2} g \left(x^2 - y^2\right) + \frac{1}{6}\left(K g + f\right)\left(x^3 - 3 x y^2\right)$$

$K = \frac{1}{R}$, $g = \frac{B_1}{B_0 R}$, $f = \frac{B_2}{B_0 R}$

- Dispersion: linear synchro-betatron coupling
- Chromaticity: terms $\sim P_u^2 \, \delta^k$
- Magnetic field terms

$\rightarrow$ Eliminate the first order synchro-betatron coupling by moving to the closed orbit system $(\hat{x}, \hat{p}_x, \hat{\sigma}, \hat{\delta})$ where all first order synchro-betatron terms $\sim \hat{x}\delta^k$ and $\sim \hat{p}_x \delta^k$ cancel; this will naturally introduce dispersion

Identification of terms

Hamiltonian

$$H = -\Gamma - \frac{x}{R}\Gamma + \frac{1}{2\Gamma}P_u^2 + \delta + \frac{1}{2}K^2 x^2 + \frac{1}{2} g \left(x^2 - y^2\right) + \frac{1}{6}\left(K\, g + f\right)\left(x^3 - 3\, x\, y^2\right)$$

- Dispersion: linear synchro-betatron coupling
- Chromaticity: terms $\sim P_u^2\,\delta^k$
- Magnetic field terms

$\longrightarrow$ Eliminate the first order synchro-betatron coupling by moving to the closed orbit system $(\hat{x}, \hat{p}_x, \hat{\sigma}, \hat{\delta})$ where all first order synchro-betatron terms $\sim \hat{x}\delta^k$ and $\sim \hat{p}_x\delta^k$ cancel; this will naturally introduce dispersion

Canonical transformation to closed orbit system (2)

Generating function

$$F(x, \hat{p}_x, \sigma, \hat{\delta}; s) = x\hat{p}_x + \sigma\hat{\delta} + \sum_{k=1}^{n} \left(X_k(s)x - P_k(s)\hat{p}_x + S_k(s)\right) \delta^k$$

$$x = \hat{x} + \sum_{k=1}^{n} P_k(s)\delta^k$$

$$p_x = \hat{p}_x + \sum_{k=1}^{n} X_k(s)\delta^k$$

Canonical transformation to closed orbit system (2)

Generating function

$$F(x, \hat{p}_x, \sigma, \hat{\delta}; s) = x\hat{p}_x + \sigma\hat{\delta} + \sum_{k=1}^{n} \left(X_k(s)x - P_k(s)\hat{p}_x + S_k(s)\right)\delta^k$$

$$\begin{aligned}
\sigma &= \hat{\sigma} - \sum_{k=1}^{n} k\left(X_k(s)\hat{x} - P_k(s)\hat{p}_x + S_k(s)\right)\delta^{k-1} \\
&\quad - \sum_{k=1}^{n}\sum_{l=1}^{m} kX_k(s)P_l(s)\delta^{k+l-1} \\
\delta &= \hat{\delta} \\
\frac{\partial F}{\partial s} &= \sum_{k=1}^{n} \left(X'_k(s)x - P'_k(s)\hat{p}_x + S'_k(s)\right)\delta^k
\end{aligned}$$

Canonical transformation to closed orbit system (3)

Hamiltonian

$$H = -\Gamma - \frac{x}{R}\Gamma + \frac{1}{2\Gamma}{p_x}^2 + \frac{1}{2}K^2 x^2 + \frac{1}{2}g\left(x^2 - y^2\right) + \frac{1}{6}\left(K\,g + f\right)\left(x^3 - 3\,x\,y^2\right) + \delta$$

$K = \frac{1}{R}$, $g = \frac{B_1}{B_0 R}$, $f = \frac{B_2}{B_0 R}$

- Insert expansions for Γ, $1/\Gamma$ and x, p_x, σ, δ
- Fix a k, collect all terms $\sim \hat{x}\delta^k$ and $\sim \hat{p}_x\delta^k$ and determine the coefficients $X_k(s)$, $P_k(s)$ and $S_k(s)$ such, that these terms vanish $\rightarrow$ Hamiltonian in closed coordinate system

First order dispersion function

$k = 1$: setting the coefficienty to vanish yields

First order dispersion function

$$X_1'(s) + (K^2 + g)P_1(s) - K = 0$$
$$-P_1'(s) + X_1(s) = 0$$

$$D''(s) + (K^2 + g)D(s) = K$$

$D(s) = P_1(s)$

Second order dispersion function

$k = 2$: setting the coefficienty to vanish yields

Second order dispersion function

$$X_2'(s) + \frac{K}{2\gamma_0^2} + (K^2 + g)P_2(s) + \frac{K}{2}X_1(s)^2 + \left(Kg + \frac{f}{2}\right)P_1(s)^2 = 0$$

$$-P_2'(s) - X_1(s) + X_2(s) + KX_1(s)P_1(s) = 0$$

$$D_2''(s) + (K^2 + g)D_2(s)$$
$$+ D''(s) - \frac{K}{2}D'(s)^2 - K^2D(s) + \left(K^3 + 2Kg + \frac{f}{2}\right)D(s)^2 = -\frac{K}{2\gamma_0^2}$$

$D(s) = P_1(s)$, $D_2(s) = P_2(s)$

Closed orbit Hamiltonian (1)

Hamiltonian up to third order in closed orbit coordinates

$$\begin{aligned} H(\hat{x}, \hat{p}_x, , \hat{y}, \hat{p}_y, \hat{\sigma}, \hat{\delta}; s) = & -\frac{1}{2}\eta' \hat{\delta}^2 + \frac{1}{3}\nu' \hat{\delta}^3 + \frac{1}{2}\hat{p}_x^2 + \frac{1}{2}\hat{p}_y^2 \\ & + \frac{1}{2} K^2 \hat{x}^2 + \frac{1}{2} g \left(\hat{x}^2 - \hat{y}^2\right) \\ & + \frac{1}{6} (K g + f) \left(\hat{x}^3 - 3 \hat{x} \hat{y}^2\right) \\ & - \frac{q}{P_0}(A_s^{\mathrm{rf}} - \bar{E}_s) \end{aligned} \quad (16)$$

Closed orbit Hamiltonian (2)

Variable annotation

$$\hat{x} = x - D\delta$$
$$\hat{p}_x = p_x - D'\delta$$
$$\hat{\sigma} = \sigma - Dp_x + D'x + DD'\delta$$
$$\hat{\delta} = \delta$$
$$\eta' = KD - \frac{1}{\gamma_0^2}$$
$$\nu' = \frac{KD}{\gamma_0^2} - \frac{1}{2}D'(s)^2 - KD_2(s) - \frac{3}{2\gamma_0^2}$$

Dynamic electric field (1)

Assumptions:

- The RF cavity to be placed in a straight section: $\frac{x}{R} = 0$
- The RF cavity is placed in a low dispersion region: $D(s) \approx 0$
- The RF field is given by $E_{\rm rf} = V_0 \delta_h(s) \sin(\omega_{\rm rf} t)$
  with $\delta_h(s)$ a periodic delta-function with the period $2\pi R$.

Then, the dynamic field term in the Hamiltonian is

$$\begin{aligned}
\frac{q}{P_0}(A_s^{\rm rf} - \bar{E}_s) &= -\frac{q}{P_0} \int \left(-\frac{\partial}{\partial t} A_s^{\rm rf} - \frac{\partial}{\partial s} V \right) \, dt \\
&= -\frac{q}{P_0} \int E_{\rm rf} \, dt \\
&= \frac{q V_0}{P_0 \omega_{\rm rf}} \left(\delta_h(s) \cos(\omega_{\rm rf} t) - \cos(\varphi_0) \right) - \frac{q V_0 \sigma}{2\pi R^2 P_0 \omega_0} \sin(\varphi_0)
\end{aligned}$$

Dynamic electric field (2)

We can decompose $\delta_h(s)\cos(\omega_{\mathrm{rf}}t)$ to

$$\delta_h(s)\cos(\omega_{\mathrm{rf}}t) = \frac{1}{4\pi R} \times \sum_{n=-\infty}^{\infty} \exp\left(-ih\frac{s}{R} + i\omega_{\mathrm{rf}}t\right) + \exp\left(-ih\frac{s}{R} - i\omega_{\mathrm{rf}}t\right)$$

Averaging the Hamiltonian (1)

Averaging over many turns, only the terms for which $h = \frac{\omega_{\mathrm{rf}}}{\omega_0}$ contribute[3], hence

$$\begin{aligned}\delta_h(s)\cos(\omega_{\mathrm{rf}}t) &\rightarrow \frac{1}{2\pi R}\cos\left(\omega_{\mathrm{rf}}t - \frac{\omega_{\mathrm{rf}}}{\omega_0}\frac{s}{R}\right)\\ &= \frac{1}{2\pi R}\cos\left(\frac{h\sigma}{R}\right)\end{aligned}$$

It follows that

$$\begin{aligned}\frac{q}{P_0}\left(A_s^{\mathrm{rf}} - \bar{E}_s\right) &= \frac{qV_0}{2\pi R P_0 \omega_{\mathrm{rf}}}\left(\cos\left(\frac{h\sigma}{R}\right) - \cos(\varphi_0)\right) - \frac{qV_0\sigma}{2\pi R^2 P_0\omega_0}\sin(\varphi_0)\\ &= \frac{qV_0}{2\pi R P_0\omega_0 h}\left(\cos(\varphi - \varphi_0) - \cos(\varphi_0) - (\varphi - \varphi_0)\sin(\varphi_0)\right)\end{aligned}$$

with $\varphi = hs/R$.

[3] $\omega_0 = \frac{\beta_0 c}{R}$ the revolution frequency

Averaging the Hamiltonian (2)

In addition the variables we encountered earlier become

$$\begin{aligned}
\eta &= \frac{1}{C}\int \eta'\,ds = \frac{1}{C}\int \frac{D(s)}{R}\,ds - \frac{1}{\gamma_0^2} \\
&= \alpha - \frac{1}{\gamma_0^2} = \frac{1}{\gamma_T^2} - \frac{1}{\gamma_0^2} \qquad (17) \\
\nu &= \frac{1}{C}\int \nu'\,ds = \frac{1}{C}\int \left(\frac{D(s)}{R\gamma_0^2} - \frac{1}{2}D'(s)^2 - \frac{D_2(s)}{R}\right)\,ds - \frac{3}{2\gamma_0^2} \\
&= \beta - \frac{3}{2\gamma_0^2} \qquad (18)
\end{aligned}$$

η and ν the first and the second order slippage factors.

α and β the first and the second order momentum compaction factors.

The final decoupled synchro-betatron Hamiltonian

Synchro-betatron Hamiltonian

$$
\begin{aligned}
&H(\hat{x}, \hat{p}_x, \hat{y}, \hat{p}_y, \sigma, \delta; s) = \\
&-\frac{1}{2}\eta\delta^2 + \frac{1}{3}\nu\delta^3 \\
&-\frac{qV_0}{2\pi R P_0 \omega_{\mathrm{rf}}}\left(\cos\left(\frac{h\sigma}{R}\right) - \cos(\varphi_0)\right) - \frac{qV_0\sigma}{2\pi R^2 P_0 \omega_0}\sin(\varphi_0) \\
&+\frac{1}{2}\hat{p}_x^2 + \frac{1}{2}\hat{p}_y^2 + \frac{1}{2}K^2 x^2 + \frac{1}{2} g \left(\hat{x}^2 - \hat{y}^2\right) \\
&+\frac{1}{6}(K g + f)\left(\hat{x}^3 - 3\,\hat{x}\,\hat{y}^2\right)
\end{aligned}
\tag{19}
$$

The final decoupled synchro-betatron Hamiltonian

Synchro-betatron Hamiltonian

$$
\begin{aligned}
&H(\hat{x}, \hat{p}_x, \hat{y}, \hat{p}_y, \sigma, \delta; s) = \\
&-\frac{1}{2}\eta\delta^2 + \frac{1}{3}\nu\delta^3 \\
&-\frac{qV_0}{2\pi R P_0 \omega_{\mathrm{rf}}}\left(\cos\left(\frac{h\sigma}{R}\right) - \cos(\varphi_0)\right) - \frac{qV_0\sigma}{2\pi R^2 P_0 \omega_0}\sin(\varphi_0) \\
&+\frac{1}{2}\hat{p}_x^2 + \frac{1}{2}\hat{p}_y^2 + \frac{1}{2}K^2\,x^2 + \frac{1}{2}g\,\left(\hat{x}^2 - \hat{y}^2\right) \\
&+\frac{1}{6}\,(K\,g + f)\,\left(\hat{x}^3 - 3\,\hat{x}\,\hat{y}^2\right)
\end{aligned}
\tag{19}
$$

Longitudinal Hamiltonian

We relabel

$q \to -e\,, \quad V_0 \to V_m\,, \quad P_0 \to p_0\,, \quad \sigma \to \zeta.$

Should look familiar...

Longitudinal Hamiltonian

$$H(s) = -\frac{1}{2}\eta\delta^2 - \frac{qV_0}{2\pi R P_0 \omega_{\mathrm{rf}}}\left(\cos\left(\frac{h\sigma}{R}\right) - \cos(\varphi_0)\right) - \frac{qV_0\sigma}{2\pi R^2 P_0 \omega_0}\sin(\varphi_0)$$

$$H(t) = -\frac{1}{2}\beta c\eta\delta^2 + \frac{eV_m}{2\pi h p_0}\left(\cos\left(\frac{h\zeta}{R}\right) - \cos(\Phi_s)\right) + \frac{eV_m\sin(\Phi_s)}{p_0 C}\zeta$$

$h = \frac{\omega_{\mathrm{rf}}}{\omega_0}\,, \quad \omega_0 = \frac{\beta c}{R}$

Longitudinal Hamiltonian

Equations of longitudinal dynamics

Some times the bunch is made to sit in an **accelerating bucket** only to compensate for external losses

- in a lepton storage ring, to compensate for synchrotron radiation losses
- in general, a bunch in a stationary bucket can move to a synchronous phase different from 0 or π in order to compensate for impedance losses (see further)

$$\begin{cases} \dfrac{d\zeta}{dt} = -\eta\beta c\delta \\ \dfrac{d\delta}{dt} = \dfrac{eV_m}{p_0 C}\left[\sin\left(\dfrac{h\zeta}{R} + \Phi_s\right) - \sin\Phi_s\right] \end{cases}$$

$$H = -\frac{1}{2}\beta c\eta\delta^2 + \frac{eV_m}{2\pi h p_0}\cos\left(\frac{h\zeta}{R}\right) + \frac{eV_m \sin\Phi_s}{p_0 C}\zeta$$

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Longitudinal Hamiltonian

We relabel

$$q \rightarrow -e\,, \quad V_0 \rightarrow V_m\,, \quad P_0 \rightarrow p_0\,, \quad \sigma \rightarrow \zeta.$$

Should look familiar... we have recovered Giovanni's formula!

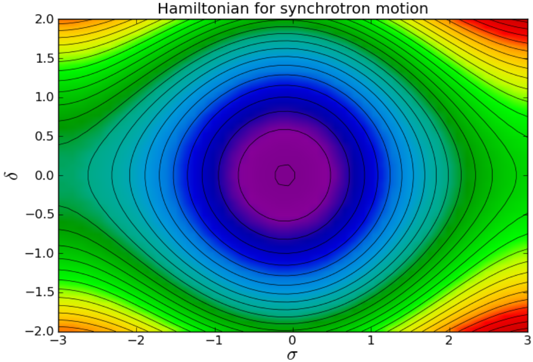
Longitudinal Hamiltonian

$$H(s) = -\frac{1}{2}\eta\delta^2 - \frac{qV_0}{2\pi R P_0 \omega_{\mathrm{rf}}}\left(\cos\left(\frac{h\sigma}{R}\right) - \cos(\varphi_0)\right) - \frac{qV_0\sigma}{2\pi R^2 P_0 \omega_0}\sin(\varphi_0)$$

$$H(t) = -\frac{1}{2}\beta c\eta\delta^2 + \frac{eV_m}{2\pi h p_0}\left(\cos\left(\frac{h\zeta}{R}\right) - \cos(\Phi_s)\right) + \frac{eV_m\sin(\Phi_s)}{p_0 C}\zeta$$

$h = \frac{\omega_{\mathrm{rf}}}{\omega_0}\,, \quad \omega_0 = \frac{\beta c}{R}$

Longitudinal action



Synchrotron tune

$$Q_s = \sqrt{\frac{qV_0\eta h}{2\pi E_0\beta_0^2}\cos(\varphi_0)}$$

Conclusions

Summary

A hopefully comprehensive overview of why and how to derive a general practical Hamiltonian for circular accelerators starting from the most basic first principles

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