Synchro-Betatron Motion in Circular Accelerators

Kevin Li

May 11, 2011

Outline of Part I

1. Motivation
   - Collective effects in the longitudinal plane
   - Numerical and computational tools in accelerator physics
2. Basic Model
   - Basic physics
   - Specialisation to classical electromagnetic theory
3. Basic Dynamics
   - The symplectic structure

Outline of Part II

4. The Transverse Hamiltonian
   - Canonical transformations
   - Coordinate and rescaling transformation
   - Transverse dynamics

Outline of Part III

5. The Synchro-Betatron Hamiltonian
   - Series of canonical transformations
   - RF fields
   - The full Synchro-Betatron Hamiltonian

Part II

The Transverse Hamiltonian

Summary of Part I

Summary

- We created our universe by declaring two manifolds and connecting them with a map[a]
- We introduced physics by postulating the principle of least action
- We defined the Lagrangian via the action
- We defined the Hamiltonain as the Legendre transform of the Lagrangian
- We then applied the principle of least action and obtained the Hamilton equations of motion which revealed the full symplectic structure of a dynamical system

[a] This allowed us to define the action

Outline

4 The Transverse Hamiltonian
- Canonical transformations
- Coordinate and rescaling transformation
- Transverse dynamics

Canonical transformations

We start from the general electromagnetic Hamiltonian

$$H(q, P, t) = \sqrt{(\vec{P} - q\vec{A})^2 c^2 + m^2 c^4} + qV$$

This Hamiltonian is still very general and not particularly useful in practice:

- What is the coordinate system?
- What are $\vec{A}$ and V?

Canonical transformations

We start from the general electromagnetic Hamiltonian

$$H(q,P,t) = \sqrt{(\vec{P} - q\vec{A})^2 c^2 + m^2 c^4} + qV$$

This Hamiltonian is still very general and not particularly useful in practice:

- What is the coordinate system?
- What are $\vec{A}$ and V?

Canonical transformations

On the next couple of slides we will derive the practical Hamiltonian up to third order; it will evolve quite naturally, we will not need to do a lot of thinking

$$
\begin{aligned}
&H(\hat{x}, \hat{p}_x, \hat{y}, \hat{p}_y, \sigma, \delta; s) = \\
&-\frac{1}{2}\eta\delta^2 + \frac{1}{3}\nu\delta^3 \\
&-\frac{qV_0}{2\pi R P_0 \omega_{\mathrm{rf}}}\left(\cos\left(\frac{h\sigma}{R}\right) - \cos(\varphi_0)\right) - \frac{qV_0\sigma}{2\pi R^2 P_0 \omega_0}\sin(\varphi_0) \\
&+\frac{1}{2}\hat{p}_x^2 + \frac{1}{2}\hat{p}_y^2 + \frac{1}{2}\left(K^2 + g\right)\hat{x}^2 - \frac{1}{2}g\hat{y}^2 \\
&+\frac{1}{6}Kg\hat{x}^3 + \frac{1}{6}f\hat{x}^3 - \frac{1}{2}Kg\hat{x}\hat{y}^2 - \frac{1}{2}f\hat{x}\hat{y}^2
\end{aligned}
$$

We will reach this Hamiltonian with a series of canonical transformations.

Why canonical transformations?

- The Hamilton equations of motion arised out of the principle of least action from the fact that the Hamiltonian is the Legendre transform of the Lagrangian
- The Hamiltonian is thus a function of two independent sets of variables: the generalized coordinates and the corresponding canonical momenta
- An arbitrary transformation of these two sets of variables in general will break the canonical conjugate relationship between them and will not render the Hamilton equations of motion forminvariant, i.e.
  - $\rightarrow$ will not preserve the symplectic structure
  - $\rightarrow$ will violate the principle of least action
  - $\rightarrow$ will lead to a description of dynamics which is non-physical

$\rightarrow$ We require a simultaneous transformation of q and P, i.e. a transformation in phase space

Why canonical transformations?

- The Hamilton equations of motion arised out of the principle of least action from the fact that the Hamiltonian is the Legendre transform of the Lagrangian
- The Hamiltonian is thus a function of two independent sets of variables: the generalized coordinates and the corresponding canonical momenta
- An arbitrary transformation of these two sets of variables in general will break the canonical conjugate relationship between them and will not render the Hamilton equations of motion forminvariant, i.e.
  - $\rightarrow$ will not preserve the symplectic structure
  - $\rightarrow$ will violate the principle of least action
  - $\rightarrow$ will lead to a description of dynamics which is non-physical

$\rightarrow$ We require a simultaneous transformation of q and P, i.e. a transformation in phase space

Why canonical transformations?

- The Hamilton equations of motion arised out of the principle of least action from the fact that the Hamiltonian is the Legendre transform of the Lagrangian
- The Hamiltonian is thus a function of two independent sets of variables: the generalized coordinates and the corresponding canonical momenta
- An arbitrary transformation of these two sets of variables in general will break the canonical conjugate relationship between them and will not render the Hamilton equations of motion forminvariant, i.e.
  - $\rightarrow$ will not preserve the symplectic structure
  - $\rightarrow$ will violate the principle of least action
  - $\rightarrow$ will lead to a description of dynamics which is non-physical

$\rightarrow$ We require a simultaneous transformation of q and P, i.e. a transformation in phase space

Why canonical transformations?

- Transformation from old variables (q, p) to new variables (Q, P):

$$Q_i = Q_i(q, p, t)\,, \quad P_i = P_i(q, p, t)$$

- The new variables must be canonically conjugate:

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}\,, \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i}$$

- The Hamilton equations of motion must hold identically for both cases:

$$\delta \int_0^1 p_i\,dq_i - H(q, p, t)\,dt = 0$$

$$\delta \int_0^1 P_i\,dQ_i - K(Q, P, t)\,dt = 0$$

$$\Leftrightarrow \quad p_i\,dq_i - P_i\,dQ_i - \left(H(q, p, t) - K(Q, P, t)\right)\,dt = \frac{dF}{dt}\,, \quad \delta F|_0^1 = 0$$

The generating function

F is called the generating function. It is a function of phase space coordinates and as such it's variation vanishes at the end points.

Assume $F = F_1(q, Q, t)$,
then

$$\begin{aligned} \frac{dF_1}{dt} &= \frac{\partial F_1}{\partial q_i}\, dq_i + \frac{\partial F_1}{\partial Q_i}\, dQ_i + \frac{\partial F_1}{\partial t}\, dt \\ &= p_i\, dq_i - P_i\, dQ_i - \left(H(q,p,t) - K(Q,P,t)\right)\, dt \end{aligned}$$

and hence

$$p_i = \frac{\partial F_1}{\partial q_i}\,, \quad P_i = -\frac{\partial F_1}{\partial Q_i}\,, \quad K(Q,P,t) = H + \frac{\partial F_1}{\partial t}$$

Other generating functions

1. $F_1(q, Q, t) \to q = q(q, Q, t)\,, \quad p = p(q, Q, t)$

$$p_i = \frac{\partial F_1}{\partial q^i}\,, \quad P_i = -\frac{\partial F_1}{\partial Q^i}\,, \quad K = H + \frac{\partial F_1}{\partial t}$$

2. $F_2(q, P, t) \to q = q(q, P, t)\,, \quad p = p(q, P, t)$

$$p_i = \frac{dF_2}{dq^i}\,, \quad Q^i = \frac{dF_2}{dP_i}\,, \quad K = H + \frac{\partial F_2}{\partial t}$$

3. $F_3(Q, p, t) \to q = q(Q, p, t)\,, \quad p = p(Q, p, t)$

$$q^i = -\frac{\partial F_3}{\partial p_i}\,, \quad P_i = -\frac{\partial F_3}{\partial Q^i}\,, \quad K = H + \frac{\partial F_3}{\partial t}$$

4. $F_4(p, P, t) \to q = q(p, P, t)\,, \quad p = p(p, P, t)$

$$q^i = -\frac{\partial F_4}{\partial p_i}\,, \quad Q_i = \frac{\partial F_4}{\partial P^i}\,, \quad K = H + \frac{\partial F_4}{\partial t}$$

Practical usage of a generating function

- Suppose we have a Hamiltonian $H = H(q, p)$
- We have found some nice coordinates $Q(q, p)$ to which we would like to transform
- We need to find the corresponding canonically conjugate variables P and the new Hamiltonian $K(Q, P)$

Practical usage of a generating function

- Suppose we have a Hamiltonian $H = H(q, p)$
- We have found some nice coordinates $Q(q, p)$ to which we would like to transform
- We need to find the corresponding canonically conjugate variables P and the new Hamiltonian $K(Q, P)$

The transformation can be generated by a function $F_3(Q, p)$:

1. Invert $Q(q, p) \to q(Q, p)$
2. $q_i = -\dfrac{\partial F_3}{\partial p^i}\,, \quad P^i = -\dfrac{\partial F_3}{\partial Q_i}\,, \quad K = H + \dfrac{\partial F_3}{\partial t}$
3. $F_3(Q, p) = -\displaystyle\int q(Q, p)\, dp\,, \quad H(q, p) \to K(Q, P)$

Practical usage of a generating function

- Suppose we have a Hamiltonian $H = H(q, p)$
- We have found some nice coordinates $Q(q, p)$ to which we would like to transform
- We need to find the corresponding canonically conjugate variables P and the new Hamiltonian $K(Q, P)$

The transformation can be generated by a function $F_3(Q, p)$:

1. Invert $Q(q, p) \to q(Q, p)$
2. $q_i = -\dfrac{\partial F_3}{\partial p^i}\,, \quad P^i = -\dfrac{\partial F_3}{\partial Q_i}\,, \quad K = H + \dfrac{\partial F_3}{\partial t}$
3. $F_3(Q, p) = -\displaystyle\int q(Q, p)\, dp\,, \quad H(q, p) \to K(Q, P)$

Practical usage of a generating function

- Suppose we have a Hamiltonian $H = H(q, p)$
- We have found some nice coordinates $Q(q, p)$ to which we would like to transform
- We need to find the corresponding canonically conjugate variables P and the new Hamiltonian $K(Q, P)$

The transformation can be generated by a function $F_3(Q, p)$:

1. Invert $Q(q, p) \to q(Q, p)$
2. $q_i = -\dfrac{\partial F_3}{\partial p^i}\,, \quad P^i = -\dfrac{\partial F_3}{\partial Q_i}\,, \quad K = H + \dfrac{\partial F_3}{\partial t}$
3. $F_3(Q, p) = -\int q(Q, p)\, dp\,, \quad H(q, p) \to K(Q, P)$

Practical usage of a generating function

- Suppose we have a Hamiltonian $H = H(q, p)$
- We have found some nice coordinates $Q(q, p)$ to which we would like to transform
- We need to find the corresponding canonically conjugate variables P and the new Hamiltonian $K(Q, P)$

The transformation can be generated by a function $F_3(Q, p)$:

1. Invert $Q(q, p) \to q(Q, p)$
2. $q_i = -\dfrac{\partial F_3}{\partial p^i}\,, \quad P^i = -\dfrac{\partial F_3}{\partial Q_i}\,, \quad K = H + \dfrac{\partial F_3}{\partial t}$
3. $F_3(Q, p) = -\displaystyle\int q(Q, p)\, dp\,, \quad H(q, p) \to K(Q, P)$

Spatial translation and Frenet-Serret coordinates

General Hamiltonian

$$H_0 = \sqrt{(\vec{P} - q\vec{A})^2 c^2 + m^2 c^4} + qV \quad (2)$$

Coordinate transformation

- Parameter transformation: $t \to s\,,\ H \to -P_s = H$
- Coordinate transformation: Frenet-Serret

$$H(x, P_x, y, P_y, t, H_0; s) = -(1 + \frac{x}{R}) \times \sqrt{\frac{(H_0 - qV)^2}{c^2} - (P_x - qA_x)^2 - (P_y - qA_y)^2 - m^2 c^2} - q\left(1 + \frac{x}{R}\right) A_s \quad (3)$$

Scaling transformation (1)

From now on we set

$$(x, y) \to u\,, \quad (P_x, P_y) \to P_u\,, \quad (A_x, A_y) \to A_u$$

With

$$P^2 = \frac{(H_0 - qV)^2}{c^2} - m^2c^2$$

we can write

Hamiltonian

$$H(u, P_u, t, H_0; s) = -\left(1 + \frac{x}{R}\right)\sqrt{P^2 - (P_u - qA_u)^2} - q\left(1 + \frac{x}{R}\right)A_s \tag{4}$$

Scaling transformation (2)

- We want to rescale $\tilde{H} \to \frac{H}{P_0}$ with $P_0 = \frac{E_0\beta_0}{c}$
- For this we need to rescale the canonically conjugate variables accordingly so that $\tilde{u}\tilde{P}_u = \frac{uP_u}{P_0}\,,\; s_0 H_s = \frac{tH_0}{P_0}$

Scaling transformation

- $\tilde{H} \to \frac{H}{P_0} \Rightarrow \tilde{u} \to u\,, \tilde{P}_u \to \frac{P_u}{P_0}\,,\; s_0 \to \beta_0 ct\,, H_s \to \frac{H_0}{E_0\beta_0^2}$
- We immediately relabel all "∼"-variables and obtain:

$$H(u, P_u, s_0, H_s; s) = -\left(1 + \frac{x}{R}\right) \times \sqrt{\frac{P^2}{P_0^2} - \left(P_u - \frac{q}{P_0}A_u\right)^2} - \frac{q}{P_0}\left(1 + \frac{x}{R}\right)A_s \tag{5}$$

Magnetic rigidity

With the magnetic rigidity of the reference orbit $\frac{P_0}{q} = B_0 R$ and with $\frac{1}{B_0 R} A_u \ll P_u$ we can finally write [2]

Hamiltonian

$$H(u, P_u, s_0, H_s; s) = -\left(1 + \frac{x}{R}\right)\sqrt{\frac{P^2}{P_0^2} - P_u^2} - \frac{1}{B_0 R}\left(1 + \frac{x}{R}\right) A_s \tag{6}$$

[2] $R = 1/K$: ring radius

Transverse expansion

With $P \approx P_0$ and $P_u \ll 1$ we can expand the square root to obtain

$$H = -\left(1 + \frac{x}{R}\right)\frac{P}{P_0} + \frac{1}{2}\left(1 + \frac{x}{R}\right)\frac{P_0}{P}P_u^2 - \frac{1}{B_0 R}\left(1 + \frac{x}{R}\right)A_s \quad (7)$$

We can see how the synchro-betatron coupling vanishes for $P = P_0$ where we obtain

$$H = -\left(1 + \frac{x}{R}\right) + \frac{1}{2}\left(1 + \frac{x}{R}\right)P_u^2 - \frac{1}{B_0 R}\left(1 + \frac{x}{R}\right)A_s \quad (8)$$

Vector potential expansion

The vector potential expansion can be obtained directly from Laplace's equation using the standard method of coefficient recursion after a polynomial Ansatz. For a normal magnet, neglecting fringe fields, the expansion becomes

$$
\begin{aligned}
A_s = &-B_0 \left(x - \frac{1}{2R} x^2 + \frac{1}{2R^2} x^3 - \frac{1}{2R^3} x^4 \right) \\
&- B_1 \left(\frac{x^2 - y^2}{2} - \frac{1}{6R} x^3 + \frac{4x^4 - y^4}{24R^2} \right) \\
&- B_2 \left(\frac{x^3 - 3xy^2}{6} - \frac{x^4 - y^4}{24R} \right) \\
&- B_3 \left(\frac{x^4 - 6x^2 y^2 + y^4}{24} \right)
\end{aligned}
$$

Transverse Hamiltonian

Inserting the vector potential expansion we obtain the transverse Hamiltonian up to third order in x, y

Transverse Hamiltonian

$$
\begin{aligned}
H(x, P_x, y, P_y) = & \frac{1}{2} P_u^2 + \frac{1}{2}\left(K^2 + g\right) x^2 - \frac{1}{2} g y^2 \\
& + \frac{1}{6} K g x^3 + \frac{1}{6} f x^3 - \frac{1}{2} K g x y^2 - \frac{1}{2} f x y^2
\end{aligned}
$$

$K = \frac{1}{R}$, $g = \frac{B_1}{B_0 R}$, $f = \frac{B_2}{B_0 R}$

Transverse Hamiltonian

Inserting the vector potential expansion we obtain the transverse Hamiltonian up to third order in x, y