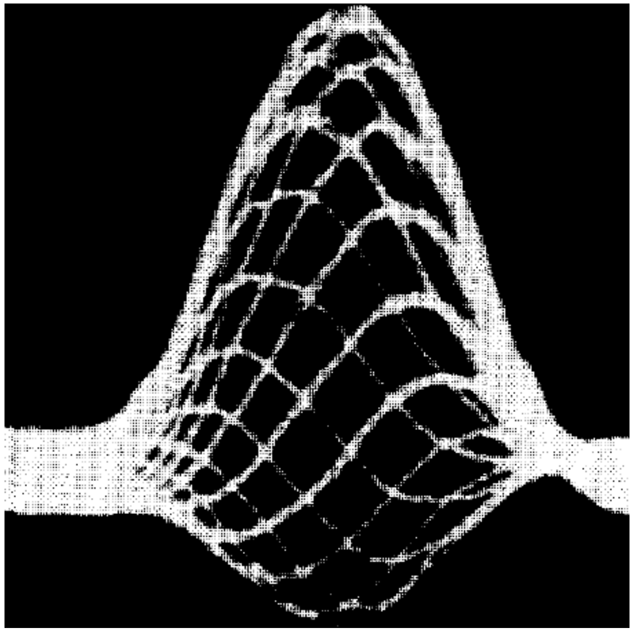
TRANSVERSE INSTABILITIES

E. Métral (CERN)

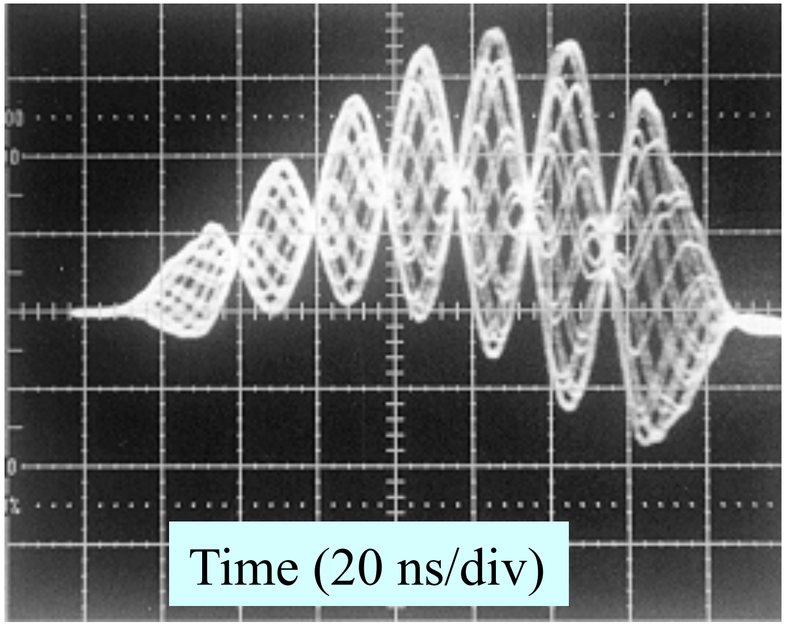
➔ The purpose of this course is to explain (theoretically) such pictures of "transverse (single-bunch) instability"

Following Laclare (and longitudinal)

Observation in the CERN PSB in ~1974 (J. Gareyte and F. Sacherer)



Observation in the CERN PS in 1999



SINGLE PARTICLE TRANSVERSE MOTION (1/3)

- A purely linear synchrotron oscillation around the synchronous particle is assumed (with no coherent oscillations)

$$\ddot{\tau} + \omega_s^2 \, \tau = 0 \qquad \qquad \tau = \hat{\tau} \cos\left(\omega_s \, t + \psi_0 \right)$$

- For the transverse betatron oscillation, the equation of unperturbed motion, e.g. in the horizontal plane, is written as

$$\boxed{x = \hat{x} \cos\left[\varphi_x \left(t \right) \right]} \qquad \qquad x^2 + \frac{\dot{x}^2}{\dot{\varphi}_x^2} = \hat{x}^2$$

- The horizontal betatron frequency is given by $\dot{\varphi}_x = Q_x \, \Omega$

Chromaticity

with $$\xi_x = \frac{\Delta Q_x \, / \, Q_{x0}}{\Delta p \, / \, p_0} \qquad \qquad \eta = - \frac{\Delta \Omega \, / \, \Omega_0}{\Delta p \, / \, p_0} = \frac{\dot{\tau}}{\Delta p \, / \, p_0}$$

SINGLE PARTICLE TRANSVERSE MOTION (2/3)

$$\Rightarrow \qquad Q_x(p) = Q_{x0}\left(1 + \xi_x \frac{\Delta p}{p_0}\right) \qquad \Omega(p) = \Omega_0\left(1 - \eta \frac{\Delta p}{p_0}\right)$$

$$\Rightarrow \qquad \dot{\varphi}_x = Q_x\,\Omega \approx Q_{x0}\,\Omega_0\left[1 - \dot{\tau}\left(1 - \frac{\xi_x}{\eta}\right)\right]$$

and

$$\boxed{\varphi_x = Q_{x0}\,\Omega_0\,(t - \tau) + \omega_{\xi_x}\,\tau + \varphi_{x0}}$$

$$\boxed{\omega_{\xi_x} = Q_{x0}\,\Omega_0\,\frac{\xi_x}{\eta}}$$

Horiz. chromatic frequency

SINGLE PARTICLE TRANSVERSE MOTION (3/3)

- In the absence of perturbation, the horizontal coordinate satisfies

$$\ddot{x} - \frac{\ddot{\varphi}_x}{\dot{\varphi}_x}\,\dot{x} + \dot{\varphi}_x^2\, x = 0$$

- In the presence of electromagnetic fields induced by the beam, the equation of motion writes

$$\ddot{x} - \frac{\ddot{\varphi}_x}{\dot{\varphi}_x}\,\dot{x} + \dot{\varphi}_x^2\, x = F_x = \frac{e}{\gamma\, m_0}\left[\vec{E} + \vec{v} \times \vec{B}\right]_x \left(t\,, \vartheta = \Omega_0 \left(t - \tau\right)\right)$$

When following the particle along its trajectory

SINGLE PARTICLE TRANSVERSE SIGNAL (1/2)

- The horizontal signal induced at a perfect PU electrode (infinite bandwidth) at angular position ϑ in the ring by the off-centered test particle is given by

$$s_x(t,\vartheta) = s_z(t,\vartheta)\, x(t) = s_z(t,\vartheta)\, \hat{x} \cos(\varphi_x)$$

$$\Rightarrow \quad s_x(t,\vartheta) = e\, \hat{x} \cos(\varphi_x) \sum_{k=-\infty}^{k=+\infty} \delta\left(t - \tau - \frac{\vartheta}{\Omega_0} - \frac{2k\pi}{\Omega_0} \right)$$

- Developing $\cos(\varphi_x)$ into exponential functions and using relations given in the longitudinal course, yields

$$s_x(t,\vartheta) = \frac{e\,\Omega_0}{4\pi}\, \hat{x}\, e^{j(Q_{x0}\,\Omega_0\, t + \varphi_{x0})} \sum_{p,m=-\infty}^{p,m=+\infty} j^{-m}\, J_{m,x}(p,\hat{\tau})\, e^{j[\omega_{pm} t + m\psi_0 - p\vartheta]}$$

$+ c.c.$ — Complex conjugate

$$\omega_{pm} = p\,\Omega_0 + m\,\omega_s$$

SINGLE PARTICLE TRANSVERSE SIGNAL (2/2)

with $J_{m,x}\left(p,\hat{\tau}\right) = J_m \left\{ \left[\left(p + Q_{x0} \right) \Omega_0 - \omega_{\xi_x} \right] \hat{\tau} \right\}$ $\Rightarrow$

$$s_x\left(\omega,\vartheta\right) = \frac{e\,\Omega_0}{4\,\pi}\,\hat{x}\,e^{\,j\varphi_{x0}}$$

$$\sum_{p,m=-\infty}^{p,m=+\infty} j^{-m}\,J_{m,x}\left(p,\hat{\tau}\right)\delta\left\{\omega - \left[\left(p+Q_{x0}\right)\Omega_0 + m\,\omega_s\right]\right\} e^{\,j\left(m\psi_0 - p\vartheta\right)} + c.c.$$

- The spectrum is a line spectrum at frequencies $\left(p + Q_{x0}\right)\Omega_0 + m\,\omega_s$
- Around every betatron line $\left(p + Q_{x0}\right)\Omega_0$, there is an infinite number of **synchrotron satellites *m***
- The spectral amplitude of the mth satellite is given by $J_{m,x}\left(p,\hat{\tau}\right)$
- **The spectrum is centered at the chromatic frequency** $\omega_{\xi_x} = Q_{x0}\,\Omega_0\,\dfrac{\xi_x}{\eta}$

STATIONARY DISTRIBUTION (1/2)

- In the absence of perturbation, $\hat{x}$ and $\hat{\tau}$ are constants of the motion

- Therefore, the stationary distribution is a function of the peak amplitudes only

$$\Psi_{x0}\left(\hat{x}\,,\,\hat{\tau}\right)$$

- No correlation between horizontal and longitudinal planes is assumed and the stationary part is thus written as the product of 2 stationary distributions, one for the longitudinal phase space and one for the horizontal one

$$\boxed{\Psi_{x0}\left(\hat{x}\,,\,\hat{\tau}\right) = f_0\left(\hat{x}\right) g_0\left(\hat{\tau}\right)}$$

$$\int_{\hat{x}=0}^{\hat{x}=+\infty} f_0\left(\hat{x}\right)\hat{x}\,d\hat{x} = \frac{1}{2\pi} \qquad \int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} g_0\left(\hat{\tau}\right)\hat{\tau}\,d\hat{\tau} = \frac{1}{2\pi}$$

STATIONARY DISTRIBUTION (2/2)

- Since on average, the beam center of mass is on axis, the horizontal signal induced by the stationary distribution is null

$$S_{x0}(t,\vartheta) = N_b \int_{\hat{x}=0}^{\hat{x}=+\infty} \int_{\varphi_{x0}=0}^{\varphi_{x0}=2\pi} \int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} \int_{\psi_0=0}^{\psi_0=2\pi} f_0(\hat{x})\, g_0(\hat{\tau})\, s_x(t,\vartheta)\, \hat{x}\, \hat{\tau}\, d\hat{x}\, d\hat{\tau}\, d\varphi_{x0}\, d\psi_0$$

$$= 0$$

PERTURBATION DISTRIBUTION (1/3)

- In order to get some dipolar fields, density perturbations $\Delta\Psi_x$ that describe beam center-of-mass displacements along the bunch are assumed

- The mathematical form of the perturbations is suggested by the single-particle signal

$$s_x(t,\vartheta) = \frac{e\,\Omega_0}{4\pi}\,\hat{x} \sum_{p,m=-\infty}^{p,m=+\infty} j^{-m}\, J_{m,x}(p,\hat{\tau})\, e^{j(\varphi_{x0}+m\psi_0)}\, e^{-j\,p\vartheta}\, e^{j[(p+Q_{x0})\Omega_0+m\,\omega_s]\,t}$$

$$+\ c.c.$$

- **Low-intensity**

$$\Delta\Psi_x = h_m(\hat{x},\hat{\tau})\, e^{-j(\varphi_{x0}+m\psi_0)}\, e^{j\,\Delta\omega^x_{cm}\,t}$$

$$\Delta\omega^x_{cm} = \omega_c - m\,\omega_s << \omega_s$$

Coherent betatron frequency shift to be determined

PERTURBATION DISTRIBUTION (2/3)

- In the time domain, the horizontal signal takes the form (for a single value *m*)

$$S_x(t,\vartheta) = 2\,\pi^2 I_b \sum_{p=-\infty}^{p=+\infty} e^{-j p \vartheta}\, \sigma_{x,m}(p)\, e^{\,j[\,(p+Q_{x0})\Omega_0 + \omega_c\,]\,t}$$

Fourier transform

$$S_x(\omega,\vartheta) = 2\,\pi^2 I_b \sum_{p=-\infty}^{p=+\infty} e^{-j p \vartheta}\, \sigma_{x,m}(p)\, \delta\left\{\omega - \left[\,(p+Q_{x0})\,\Omega_0 + \omega_c\,\right]\right\}$$

with

$$\sigma_{x,m}(p) = j^{-m} \int_{\hat{x}=0}^{\hat{x}=+\infty} \int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} h_m(\hat{x},\hat{\tau})\, J_{m,x}(p,\hat{\tau})\, \hat{x}^2\, d\hat{x}\;\; \hat{\tau}\;\; d\hat{\tau}$$

PERTURBATION DISTRIBUTION (3/3)

- **High-intensity**

$$\Delta\Psi_x = \sum_m h_m\left(\hat{x}\,,\hat{\tau}\right) e^{-j\left(\varphi_{x0} + m\psi_0\right)}\; e^{\,j\,\Delta\omega_{cm}^{x}\,t}$$

TRANSVERSE IMPEDANCE

$$\left[\vec{E} + \vec{v} \times \vec{B} \right]_x (t, \vartheta) = \frac{-j \beta}{2 \pi R} \int Z_x(\omega) \, S_x(\omega, \vartheta) \, e^{j \omega t} \, d\omega$$

All the properties of the electromagnetic response of a given machine to a passing particle is gathered into the transverse impedance (complex function => in Ω / m)

EFFECT OF THE PERTURBATION (1/10)

$$\Psi_x\left(\hat{x}\,,\varphi_{x0}\,,\hat{\tau}\,,\psi_0\,,t\right)=\Psi_{x0}+\Delta\Psi_x$$

$$\Rightarrow \qquad \Psi_x = f_0\left(\hat{x}\right) g_0\left(\hat{\tau}\right) + \sum_m h_m\left(\hat{x}\,,\hat{\tau}\right) e^{-j\left(\varphi_{x0}+m\psi_0\right)}\, e^{j\,\Delta\omega_{cm}^x\,t}$$

- Vlasov equation

$$\frac{\partial\Psi_x}{\partial t}+\frac{\partial\Psi_x}{\partial\hat{x}}\dot{\hat{x}}+\frac{\partial\Psi_x}{\partial\varphi_{x0}}\dot{\varphi}_{x0}+\frac{\partial\Psi_x}{\partial\hat{\tau}}\dot{\hat{\tau}}+\frac{\partial\Psi_x}{\partial\psi_0}\dot{\psi}_0=0$$

$\Rightarrow$ **Linearized Vlasov equation**

$$\frac{\partial\Psi_x}{\partial t}=-\frac{df_0\left(\hat{x}\right)}{d\hat{x}}\,g_0\left(\hat{\tau}\right)\dot{\hat{x}}$$

$$\Rightarrow \quad j\sum_m h_m\left(\hat{x}\,,\hat{\tau}\right) e^{-j\left(\varphi_{x0}+m\psi_0\right)}\,\Delta\omega_{cm}^x\, e^{j\,\Delta\omega_{cm}^x\,t}=-\frac{df_0\left(\hat{x}\right)}{d\hat{x}}\,g_0\left(\hat{\tau}\right)\dot{\hat{x}}$$

EFFECT OF THE PERTURBATION (2/10)

- The expression of $\dot{\hat{x}}$ can be drawn from the single-particle horizontal equation of motion

$$\dot{\hat{x}} = \frac{d}{d\,t}\left(\,\hat{x}\,\right) = \frac{d}{d\,t}\left[x^2 + \left(\frac{\dot{x}}{\dot{\varphi}_x}\right)^2\right]^{1/2} = F_x \, \frac{\dot{x}}{\hat{x}\,\dot{\varphi}_x^2}$$

$$\frac{\dot{x}}{\hat{x}\,\dot{\varphi}_x} = -\sin\left(\,\varphi_x\,\right)$$

$$\Rightarrow \quad \boxed{\dot{\hat{x}} = -\frac{\sin\left(\,\varphi_x\,\right)}{\dot{\varphi}_x}\,F_x}$$

EFFECT OF THE PERTURBATION (3/10)

- Using the definition of the transverse impedance, the force can be written

$$F_x = - \frac{j \, e \, \beta \, \pi \, I_b}{R \, \gamma \, m_0} \sum_{p=-\infty}^{p=+\infty} Z_x(p) \, \sigma_{x,m}(p) \, e^{-j p \Omega_0 (t-\tau)} \, e^{j[(p+Q_{x0})\Omega_0 + \omega_c] t}$$

$$(p + Q_{x0}) \, \Omega_0 + \omega_c$$

- Developing the $\sin(\varphi_x)$ into exponential functions, keeping then only the slowly varying term, making the approximation $\dot{\varphi}_x \approx Q_{x0} \, \Omega_0$ and **using the relations** $J_{-m}(-x) = J_m(x)$ and one from the longitudinal course, yields

$$\dot{\hat{x}} = - \frac{e \, \pi \, I_b}{2 \, \gamma \, m_0 \, c \, Q_{x0}} \sum_{p,m=-\infty}^{p,m=+\infty} Z_x(p) \, \sigma_{x,m}(p) \, j^m \, J_{m,x}(p, \hat{\tau}) \, e^{-j(\varphi_{x0} + m \psi_0)} \, e^{j \Delta\omega_{cm}^x t}$$

EFFECT OF THE PERTURBATION (4/10)

$\Longrightarrow$ For each mode *m*, one has

$$j\, h_m(\hat{x}, \hat{\tau})\, \Delta\omega_{cm}^{x} = \frac{e\, \pi\, I_b}{2\, \gamma\, m_0\, c\, Q_{x0}} \sum_{p=-\infty}^{p=+\infty} Z_x(p)\, \sigma_x(p)\, j^m\, J_{m,x}(p, \hat{\tau})\, \frac{d f_0(\hat{x})}{d\hat{x}}\, g_0(\hat{\tau})$$

with

$$\boxed{\sigma_x(p) = \sum_m \sigma_{x,m}(p)}$$

Spectrum amplitude at frequency $(p + Q_{x0})\, \Omega_0 + \omega_c$

Multiplying both sides by $\hat{x}^2$ and integrating over $\hat{x}$ $\Longrightarrow$

$$j\, \Delta\omega_{cm}^{x} \int_{\hat{x}=0}^{\hat{x}=+\infty} h_m(\hat{x}, \hat{\tau})\, \hat{x}^2\, d\hat{x} = -\frac{e\, I_b}{2\, \gamma\, m_0\, c\, Q_{x0}} \sum_{p=-\infty}^{p=+\infty} Z_x(p)\, \sigma_x(p)\, j^m\, J_{m,x}(p, \hat{\tau})\, g_0(\hat{\tau})$$

using the relation $\displaystyle\int_{\hat{x}=0}^{\hat{x}=+\infty} \frac{d f_0(\hat{x})}{d\hat{x}}\, \hat{x}^2\, d\hat{x} = -2 \int_{\hat{x}=0}^{\hat{x}=+\infty} f_0(\hat{x})\, \hat{x}\, d\hat{x} = -\frac{1}{\pi}$

EFFECT OF THE PERTURBATION (5/10)

- Note that the horizontal stationary distribution disappeared and only the longitudinal one remains => Only the beam center of mass is important (in our case). This should also be valid for the perturbation, which can be written

$$\int_{\hat{x}=0}^{\hat{x}=+\infty} h_m(\hat{x}, \hat{\tau})\, \hat{x}^2\, d\hat{x} = g_0(\hat{\tau})\, \hat{x}_m(\hat{\tau})$$

Averaged peak betatron amplitude

$\Rightarrow$ **Final form of the equation of coherent motion of a single bunch:**

$$\Delta\omega_{cm}^{x} = \omega_c - m\,\omega_s$$

Contribution from all the modes *m*

$$j\,\Delta\omega_{cm}^{x}\,\hat{x}_m(\hat{\tau}) = -\frac{e\,I_b}{2\,\gamma\,m_0\,c\,Q_{x0}} \sum_{p=-\infty}^{p=+\infty} Z_x(p)\,\sigma_x(p)\,j^m\,J_{m,x}(p,\hat{\tau})$$

EFFECT OF THE PERTURBATION (6/10)

with $$\sigma_{x,m}(p) = j^{-m} \int_{\hat{x}=0}^{\hat{x}=+\infty} \int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} h_m(\hat{x},\hat{\tau})\, J_{m,x}(p,\hat{\tau})\, \hat{x}^2\, d\hat{x}\ \hat{\tau}\ d\hat{\tau}$$

$$= j^{-m} \int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} J_{m,x}(p,\hat{\tau})\, g_0(\hat{\tau})\, \hat{x}_m(\hat{\tau})\, \hat{\tau}\ d\hat{\tau}$$

- Coherent modes of oscillation **at low intensity** (i.e. considering only a single mode *m*)

$$j\, \Delta\omega_{cm}^{x}\, \hat{x}_m(\hat{\tau}) = -\frac{e\, I_b}{2\,\gamma\, m_0\, c\, Q_{x0}} \sum_{p=-\infty}^{p=+\infty} Z_x(p)\, \sigma_{x,m}(p)\, j^m\, J_{m,x}(p,\hat{\tau})$$

Multiplying both sides by $j^{-m}\, J_{m,x}(l,\hat{\tau})\, g_0(\hat{\tau})\, \hat{\tau}$ and integrating over $\hat{\tau}$

EFFECT OF THE PERTURBATION (7/10)

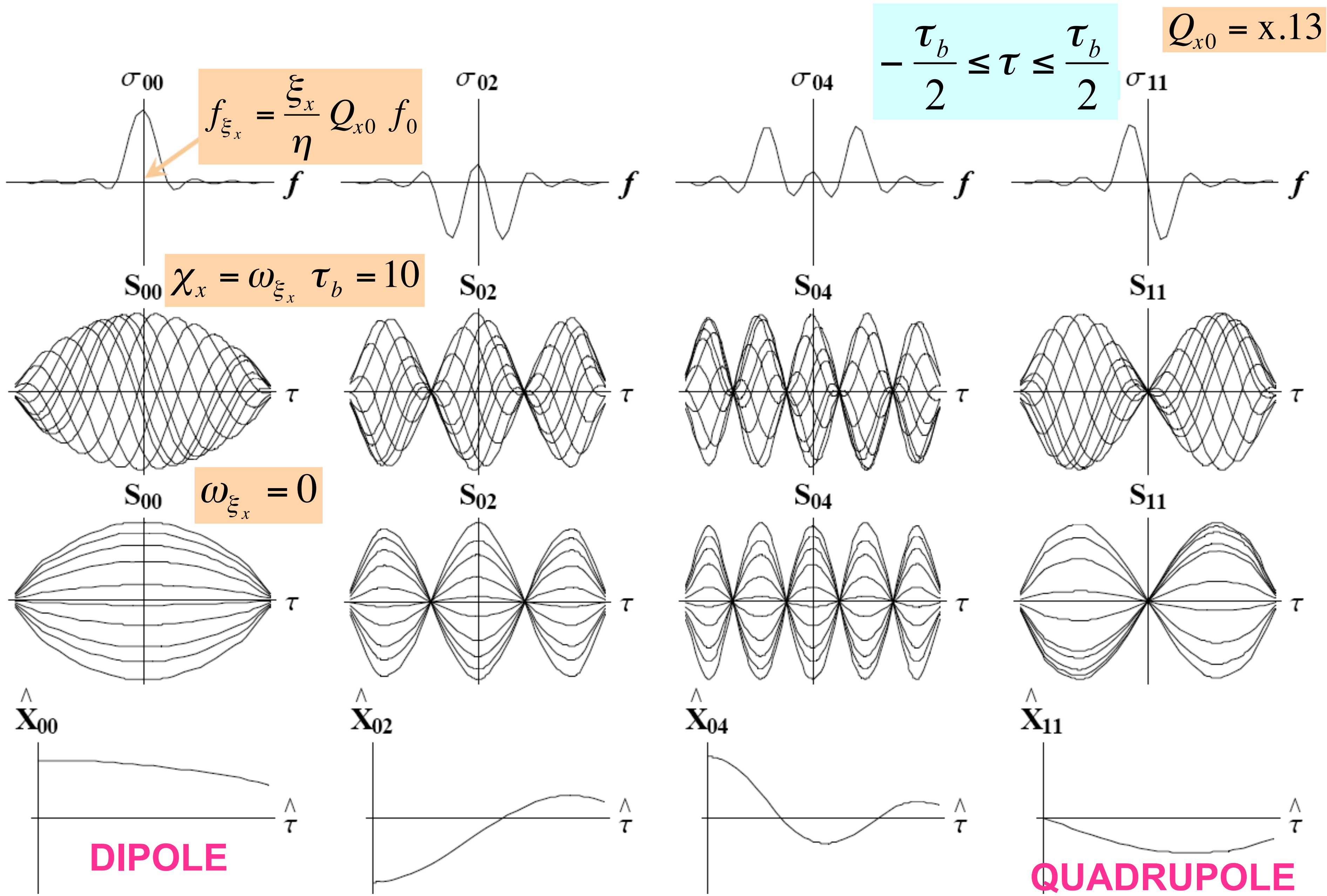
$$\Rightarrow \quad \Delta\omega^{x}_{cm}\, \sigma_{x,m}(l) = \sum_{p=-\infty}^{p=+\infty} K^{x,m}_{lp}\, \sigma_{x,m}(p)$$

$$K^{x,m}_{lp} = \frac{j\, e\, I_b}{2\,\gamma\, m_0\, c\, Q_{x0}}\, Z_x(p) \int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} J_{m,x}(l,\hat{\tau})\, J_{m,x}(p,\hat{\tau})\, g_0(\hat{\tau})\, \hat{\tau}\, d\hat{\tau}$$

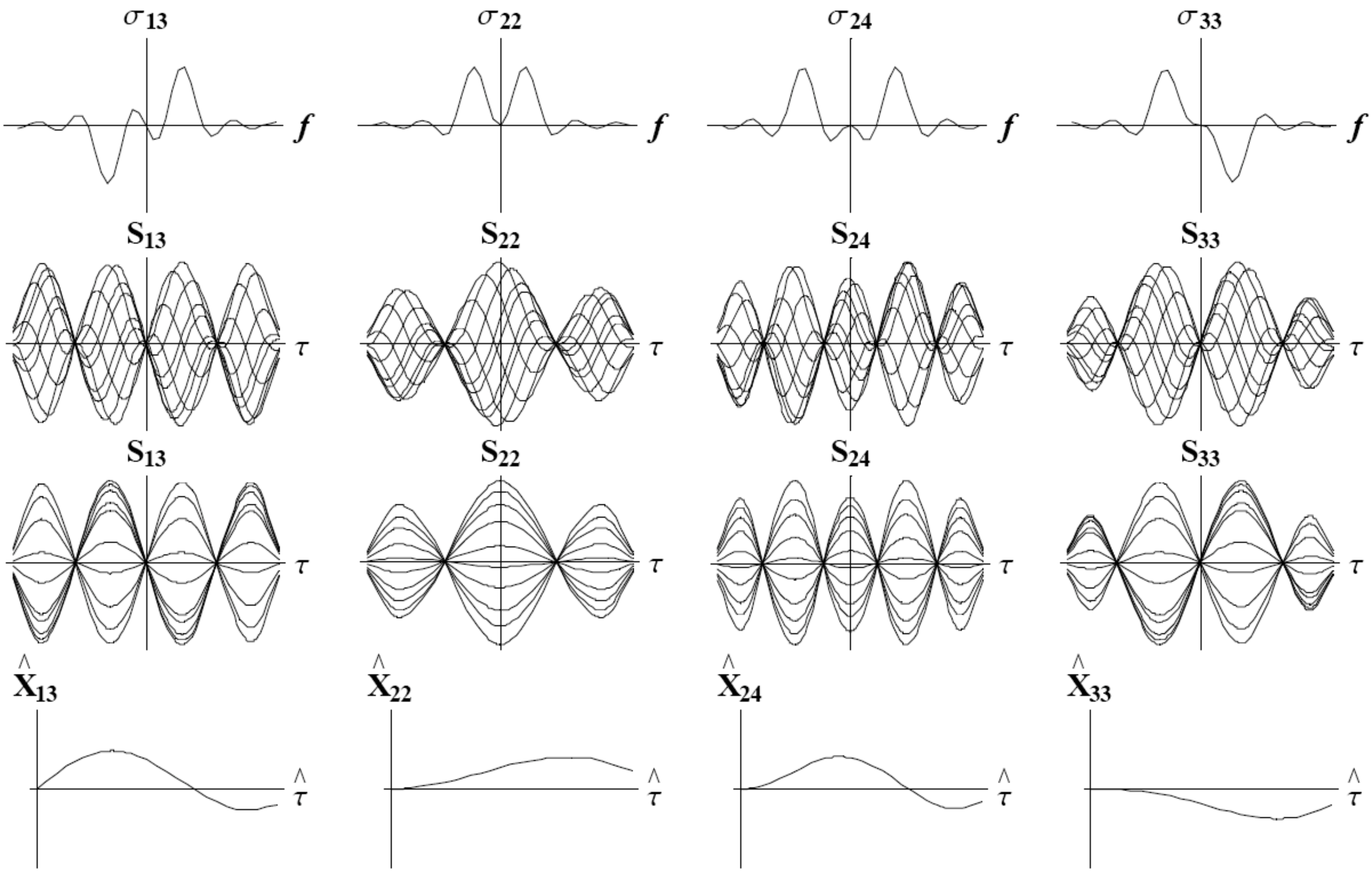
- Following the same procedure as for the longitudinal plane, the horizontal coherent oscillations (over several turns) of a **"water-bag" bunch** interacting with a constant inductive impedance are shown in the next slides for the first **head-tail modes** (Note that the index *x* has been removed for clarity)

$$g_0(\hat{\tau}) = 4 / \left(\pi\, \tau_b^2\right)$$

EFFECT OF THE PERTURBATION (8/10)

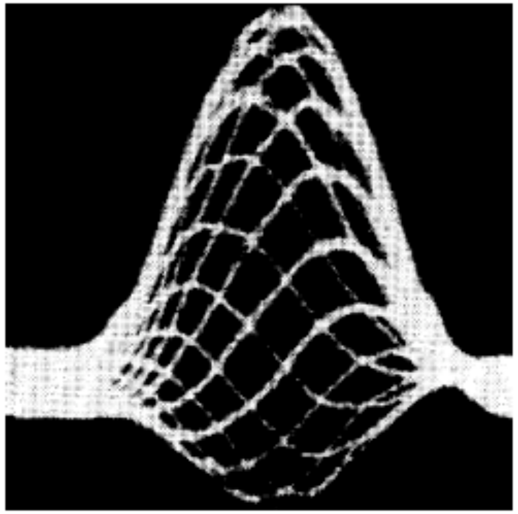


EFFECT OF THE PERTURBATION (9/10)

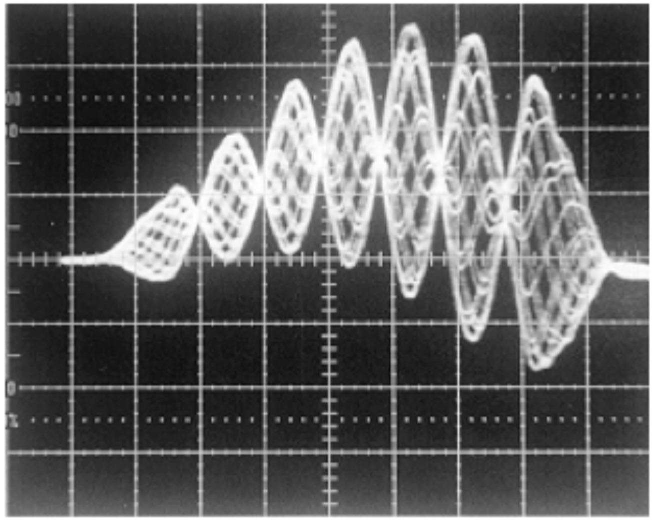


EFFECT OF THE PERTURBATION (10/10)

Observation in the CERN PSB in ~1974
(J. Gareyte and F. Sacherer)



Observation in the CERN PS in 1999



(Laclare's) theory