LONGITUDINAL INSTABILITIES

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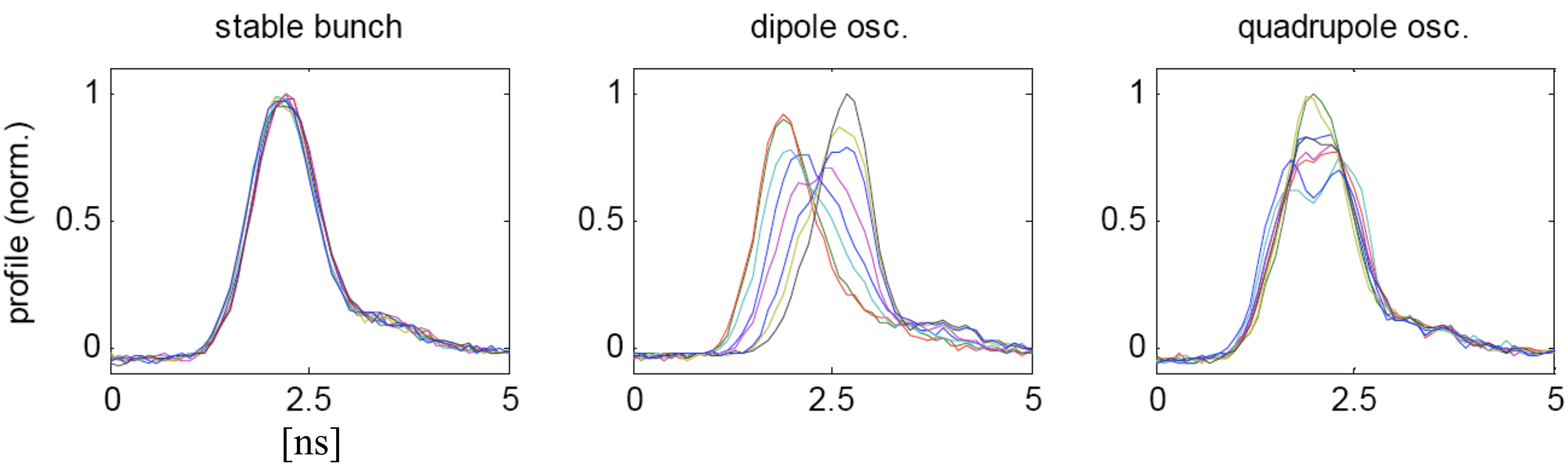
➔ The purpose of this course is to explain (theoretically) such pictures of "longitudinal (single-bunch) instability"

Following Laclare

Observations in the CERN SPS in 2007



Courtesy of Giulia Papotti

SINGLE PARTICLE LONGITUDINAL MOTION (1/2)

$$\boxed{\ddot{\tau} + \omega_{s0}^2 \, \tau = 0}$$

$$\tau = \hat{\tau} \cos\left(\omega_{s0} \, t + \psi_0 \right)$$

$$\omega_{s0} = \Omega_0 \left(- \frac{e \hat{V}_{\mathrm{RF}} \, h \, \eta \cos \phi_{s0}}{2 \pi \beta^2 \, E_{total}} \right)^{1/2}$$

Time interval between the passage of the synchronous particle and the test particle, for a fixed observer at azimuthal position ϑ

$e =$ elementary charge

$R =$ average machine radius $\quad R \, \Omega_0 = v = \beta c \quad c =$ speed of light

$p_0 =$ momentum of the synch. particle $\quad p_0 \, c = \beta \, E_{total}$

$\hat{V}_{\mathrm{RF}} =$ peak RF voltage

$$\eta = \alpha_p - \frac{1}{\gamma^2} = - \frac{\Delta f \, / \, f_0}{\Delta p \, / \, p_0} = \text{slip factor}$$

$h =$ RF harmonic number

$\phi_{s0} =$ RF phase of the synch. particle $\quad \alpha_p = \dfrac{1}{\gamma_t^2} =$ mom. comp. factor

SINGLE PARTICLE LONGITUDINAL MOTION (2/2)

- Canonical conjugate variables $\left(\tau \, , \, \dot{\tau} = \frac{d\tau}{dt} \right)$ $\quad \dot{\tau} = \frac{d\tau}{dt} = - \frac{df}{f_0} = \eta \, \frac{dp}{p_0}$

$$\tau^2 + \frac{\dot{\tau}^2}{\omega_{s0}^2} = \hat{\tau}^2$$

- Linear matching condition $\omega_{s0} = \dfrac{2 \left| \eta \right| \dfrac{\Delta p}{p_0}}{\tau_b}$ $\quad \tau_b = 2 \, \hat{\tau}_{\max}$

- Effect of the (beam-induced) electromagnetic fields $\dot{\tau} = \eta \, \dfrac{p - p_0}{p_0} \quad \Rightarrow$

$$\boxed{\ddot{\tau} + \omega_{s0}^2 \, \tau = \frac{\eta}{p_0} \frac{dp}{dt} = \frac{\eta \, e}{p_0} \left[\vec{E} + \vec{v} \times \vec{B} \right]_z \left(t \, , \vartheta = \Omega_0 \left(t - \tau \right) \right)}$$

When following the particle along its trajectory

SINGLE PARTICLE LONGITUDINAL SIGNAL (1/3)

- At time $t = 0$, the synchronous particle starts from $\vartheta = 0$ and reaches the Pick-Up (PU) electrode (assuming infinite bandwidth) at times t_k^0

$$\Omega_0 \, t_k^0 = \vartheta + 2 \, k \, \pi, \qquad - \infty \leq k \leq + \infty$$

- The test particle is delayed by τ . It goes through the electrode at times t_k

$$t_k = t_k^0 + \tau$$

- The current signal induced by the test particle is a series of impulses delivered on each passage

$$\boxed{s_z\left(t, \vartheta\right) = e \sum_{k=-\infty}^{k=+\infty} \delta\left(t - \tau - \frac{\vartheta}{\Omega_0} - \frac{2 \, k \, \pi}{\Omega_0} \right)}$$

Dirac function

SINGLE PARTICLE LONGITUDINAL SIGNAL (2/3)

- **Using the relations**

$$\boxed{\sum_{k=-\infty}^{k=+\infty} \delta\left(u - \frac{2k\pi}{\Omega_0}\right) = \frac{\Omega_0}{2\pi} \sum_{p=-\infty}^{p=+\infty} e^{j p \Omega_0 u}}$$

$$\boxed{e^{-j u \hat{\tau} \cos(\omega_{s0} t + \psi_0)} = \sum_{m=-\infty}^{m=+\infty} j^{-m} J_m(u \hat{\tau})\, e^{j m (\omega_{s0} t + \psi_0)}}$$

Bessel function of mth order

$$\Longrightarrow \quad s_z(t, \vartheta) = \frac{e\,\Omega_0}{2\pi} \sum_{p,m=-\infty}^{p,m=+\infty} j^{-m} J_m(p\,\Omega_0\,\hat{\tau})\, e^{j(\omega_{pm} t - p\vartheta + m\psi_0)}$$

$$\boxed{\omega_{pm} = p\Omega_0 + m\omega_{s0}}$$

Fourier transform

$$\boxed{s_z(\omega, \vartheta) = \frac{e\,\Omega_0}{2\pi} \sum_{p,m=-\infty}^{p,m=+\infty} j^{-m} J_m(p\,\Omega_0\,\hat{\tau})\, e^{-j(p\vartheta - m\psi_0)}\, \delta(\omega - \omega_{pm})}$$

SINGLE PARTICLE LONGITUDINAL SIGNAL (3/3)

- The single particle spectrum is a line spectrum at frequencies

$$\omega_{pm} = p\,\Omega_0 + m\,\omega_{s0}$$

- Around every harmonic of the revolution frequency $p\,\Omega_0$, there is an infinite number of **synchrotron satellites *m***

- The spectral amplitude of the mth satellite is given by $J_m\left(p\,\Omega_0\,\hat{\tau}\right)$

- The spectrum is centered at the origin

- Because the argument of the Bessel functions is proportional to $\hat{\tau}$, the width of the spectrum behaves like $\hat{\tau}^{-1}$

DISTRIBUTION OF PARTICLES (1/2)

$$\Psi\left(\hat{\tau}, \psi_0, t\right) = \text{particle density in longitudinal phase space}$$

- Signal induced (at the PU electrode) by the whole beam

$$\boxed{S_z\left(t, \vartheta\right) = N_b \int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} \int_{\psi_0=0}^{\psi_0=2\pi} \Psi\left(\hat{\tau}, \psi_0, t\right) s_z\left(t, \vartheta\right) \hat{\tau}\, d\hat{\tau}\, d\psi_0}$$

Number of particles per bunch

- Canonically conjugated variables derive from a Hamiltonian $\mathrm{H}\left(q, p, t\right)$ by the canonical equations

$$\dot{q} = \frac{\partial \mathrm{H}\left(q, p, t\right)}{\partial p} \qquad \dot{p} = -\frac{\partial \mathrm{H}\left(q, p, t\right)}{\partial q}$$

DISTRIBUTION OF PARTICLES (2/2)

- According to the **Liouville's theorem**, the particles, in a non-dissipative system of forces, move like an incompressible fluid in phase space. The constancy of the phase space density $\Psi(q, p, t)$ is expressed by the equation

$$\frac{d\Psi(q, p, t)}{d\,t} = 0$$

where the total differentiation indicates that one follows the particle while measuring the density of its immediate neighborhood. This equation, sometimes referred to as the Liouville's theorem, states that the local particle density does not vary with time when following the motion in canonical variables

- As seen by a stationary observer (like a PU electrode) which does not follow the particle => **Vlasov equation**

$$\boxed{\frac{\partial\Psi(q, p, t)}{\partial\,t} + \dot{q}\,\frac{\partial\Psi(q, p, t)}{\partial\,q} + \dot{p}\,\frac{\partial\Psi(q, p, t)}{\partial\,p} = 0}$$

STATIONARY DISTRIBUTION (1/5)

- In the case of a harmonic oscillator $\mathrm{H} = \omega \, \dfrac{q^2 + p^2}{2}$

$$\dot{q} = \frac{\partial \mathrm{H}}{\partial p} = p\,\omega$$

$$\dot{p} = -\frac{\partial \mathrm{H}}{\partial q} = -q\,\omega$$

$$\Longrightarrow \qquad \ddot{q} + \omega^2 q = 0$$

- Going to polar coordinates

$$q = r\cos\phi$$
$$p = -r\sin\phi$$

$$\Longrightarrow \qquad \frac{\partial \Psi}{\partial t} + \dot{r}\,\frac{\partial \Psi}{\partial r} + \dot{\phi}\,\frac{\partial \Psi}{\partial \phi} = 0$$

STATIONARY DISTRIBUTION (2/5)

- As r is a constant of motion $\Longrightarrow \quad \dot{r} = 0$

$$\Longrightarrow \quad \frac{\partial \Psi}{\partial t} + \omega \, \frac{\partial \Psi}{\partial \phi} = 0 \qquad \text{with} \qquad \phi = \omega \, t$$

$$\Longrightarrow \quad \frac{\partial \Psi}{\partial t} = - \, \omega \, \frac{\partial \Psi}{\partial \phi} = - \, \frac{\partial \Psi}{\partial t} \qquad \Longrightarrow \qquad \frac{\partial \Psi}{\partial t} = \frac{\partial \Psi}{\partial \phi} = 0$$

$$\Longrightarrow \quad \Psi \left(r \right)$$

A stationary distribution is any function of *r*, or equivalently any function of the Hamiltonian H

STATIONARY DISTRIBUTION (3/5)

- In our case $q = \tau \qquad r = \hat{\tau}$
  $p = \dot{\tau} \qquad \phi = \psi_0$

$$\boxed{\Psi_0\left(\hat{\tau}, \psi_0, t\right) = g_0\left(\hat{\tau}\right)}$$

$$\Rightarrow \qquad \boxed{S_{z0}\left(\omega, \vartheta\right) = 2\pi I_b \sum_{p=-\infty}^{p=+\infty} \sigma_0\left(p\right) \delta\left(\omega - p\,\Omega_0\right) e^{-j p \vartheta}}$$

$$I_b = N_b\, e\, \Omega_0 / 2\pi$$

Amplitude of the spectrum

with $$\boxed{\sigma_0\left(p\right) = \int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} J_0\left(p\,\Omega_0\,\hat{\tau}\right) g_0\left(\hat{\tau}\right) \hat{\tau}\, d\hat{\tau}}$$

STATIONARY DISTRIBUTION (4/5)

- Let's assume a parabolic amplitude density $$\boxed{g_0\left(\hat{z}\right) = \frac{2}{\pi\left(\dfrac{\tau_b}{2}\right)^2}\left(1-\hat{z}^2\right)}$$

$$\hat{z} \equiv \hat{\tau} \,/\left(\tau_b \,/\, 2\right)$$

- The line density $\lambda\left(\tau\right)$ is the projection of the distribution $g_0\left(\hat{\tau}\right)$ on the τ axis

$$\lambda\left(\tau\right) = \int g_0\left(\hat{\tau}\right) \frac{d\dot{\tau}}{\omega_{s0}}$$

$$\int \lambda\left(\tau\right) d\tau = 1 \qquad \Rightarrow \qquad \boxed{\lambda\left(z\right) = \frac{8}{3\,\pi\left(\dfrac{\tau_b}{2}\right)}\left(1-z^2\right)^{3/2}}$$

$$z \equiv \tau \,/\left(\tau_b \,/\, 2\right)$$

STATIONARY DISTRIBUTION (5/5)

- **Using the relations**

$$\int_{u'=0}^{u'=u} J_0(u')\, u'\, du' = u\, J_1(u)$$

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2\,n}{x}$$

$$\int x^3\, J_0(x)\, dx = x^2 \left[2\, J_2(x) - x\, J_3(x) \right]$$

$$\Rightarrow \quad \boxed{\sigma_0(p) = \frac{4}{\pi\,(p\,\pi\,B)^2}\, J_2(p\,\pi\,B)}$$

$$B = \tau_b\, \Omega_0 / 2\pi$$

Bunching factor

and

$$\boxed{S_{z0}(\omega, \vartheta) = 8\, I_b \sum_{p=-\infty}^{p=+\infty} \delta(\omega - p\,\Omega_0)\, e^{-j\,p\,\vartheta}\, \frac{J_2(p\,\pi\,B)}{(p\,\pi\,B)^2}}$$

LONGITUDINAL IMPEDANCE

$$\boxed{2\,\pi\,R\left[\,\vec{E}+\vec{v}\times\vec{B}\,\right]_z\left(t\,,\vartheta\,\right)=-\int_{\omega=-\infty}^{\omega=+\infty} Z_l\left(\omega\right)\,S_z\left(\omega\,,\vartheta\,\right)\,e^{\,j\,\omega\,t}\,d\omega}$$

All the properties of the electromagnetic response of a given machine to a passing particle is gathered into the impedance (complex function => in Ω)

EFFECT OF THE STATIONARY DISTRIBUTION (1/9)

$$\ddot{\tau} + \omega_{s0}^2 \, \tau = F_0 = \frac{\eta \, e}{p_0} \left[\vec{E} + \vec{v} \times \vec{B} \right]_{z0} \left(t , \vartheta = \Omega_0 \left(t - \tau \right) \right)$$

$$\left[\vec{E} + \vec{v} \times \vec{B} \right]_{z0} \left(t , \vartheta = \Omega_0 \left(t - \tau \right) \right) = - \frac{1}{2 \, \pi \, R} \int_{\omega = -\infty}^{\omega = +\infty} Z_l \left(\omega \right) S_{z0} \left(\omega , \vartheta = \Omega_0 \left(t - \tau \right) \right) e^{\, j \omega t} \, d\omega$$

$$\Rightarrow \quad \boxed{\ddot{\tau} + \omega_{s0}^2 \, \tau = F_0 = \frac{2 \, \pi \, I_b \, \omega_{s0}^2}{\Omega_0 \, \hat{V}_{\mathrm{RF}} \, h \cos \phi_{s0}} \sum_{p = -\infty}^{p = +\infty} Z_l \left(p \right) \sigma_0 \left(p \right) e^{\, j \, p \, \Omega_0 \, \tau}}$$

$p \, \Omega_0$

EFFECT OF THE STATIONARY DISTRIBUTION (2/9)

- Expanding the exponential in series (for small amplitudes)

$$\ddot{\tau} + \omega_{s0}^2 \, \tau = \frac{2 \pi \, I_b \, \omega_{s0}^2}{\Omega_0 \, \hat{V}_{\mathrm{RF}} \, h \cos \phi_{s0}} \sum_{p=-\infty}^{p=+\infty} Z_l(p) \, \sigma_0(p) \left[1 + j \, p \, \Omega_0 \, \tau - \frac{(p \, \Omega_0 \, \tau)^2}{2} + \ldots \right]$$

Synchronous phase shift

Incoherent frequency shift (potential-well distortion)

Nonlinear terms introducing some synchrotron frequency spread

EFFECT OF THE STATIONARY DISTRIBUTION (3/9)

- Synchronous phase shift

$$\ddot{\tau} + \omega_{s0}^2 \, \tau = \frac{2 \, \pi \, I_b \, \omega_{s0}^2}{\Omega_0 \, \hat{V}_{\mathrm{RF}} \, h \cos \phi_{s0}} \sum_{p=-\infty}^{p=+\infty} \mathrm{Re}\left[Z_l \left(p \right) \right] \sigma_0 \left(p \right)$$

$$\tau = t_p - t_{s0}$$

Test particle

Synchronous particle

$$\ddot{t}_{s0} = 0$$

$$\Rightarrow \quad \ddot{t}_p + \omega_{s0}^2 \, t_p = \omega_{s0}^2 \, t_{s0} + \frac{2 \, \pi \, I_b \, \omega_{s0}^2}{\Omega_0 \, \hat{V}_{\mathrm{RF}} \, h \cos \phi_{s0}} \sum_{p=-\infty}^{p=+\infty} \mathrm{Re}\left[Z_l \left(p \right) \right] \sigma_0 \left(p \right)$$

$$\Rightarrow \quad \ddot{t}_p + \omega_{s0}^2 \, t_p = \omega_{s0}^2 \, t_s$$

with $\Delta t_s = t_s - t_{s0} = \dfrac{2 \, \pi \, I_b}{\Omega_0 \, \hat{V}_{\mathrm{RF}} \, h \cos \phi_{s0}} \displaystyle\sum_{p=-\infty}^{p=+\infty} \mathrm{Re}\left[Z_l \left(p \right) \right] \sigma_0 \left(p \right)$

EFFECT OF THE STATIONARY DISTRIBUTION (4/9)

$$\phi = \omega_{\mathrm{RF}}\, t \qquad \omega_{\mathrm{RF}} = h\, \Omega_0$$

$$\phi_s = \omega_{\mathrm{RF}}\, t_s$$

$$\Delta\phi_s = \phi_s - \phi_{s0} = \omega_{\mathrm{RF}}\, \Delta t_s$$

Only for the small amplitudes. For the power loss of the whole bunch an averaging is needed!

$$\Longrightarrow \quad \boxed{\Delta\phi_s = \phi_s - \phi_{s0} = \frac{2\,\pi\, I_b}{\hat{V}_{\mathrm{RF}}\cos\phi_{s0}} \sum_{p=-\infty}^{p=+\infty} \mathrm{Re}\left[Z_l(p)\right] \sigma_0(p)}$$

Can be used to probe the resistive part of the longitudinal impedance

EFFECT OF THE STATIONARY DISTRIBUTION (5/9)

- Incoherent synchrotron frequency shift (potential-well distortion)

$$\ddot{\tau} + \omega_{s0}^2 \, \tau = \frac{2 \, \pi \, I_b \, \omega_{s0}^2}{\Omega_0 \, \hat{V}_{\mathrm{RF}} \, h \cos \phi_{s0}} \sum_{p=-\infty}^{p=+\infty} Z_l(p) \, \sigma_0(p) \, j \, p \, \Omega_0 \, \tau$$

$$\Longrightarrow \quad \ddot{\tau} + \omega_s^2 \, \tau = 0$$

with $\quad \omega_s^2 = \omega_{s0}^2 \left[1 - \dfrac{2 \, \pi \, I_b}{\hat{V}_{\mathrm{RF}} \, h \cos \phi_{s0}} \displaystyle\sum_{p=-\infty}^{p=+\infty} j \, Z_l(p) \, p \, \sigma_0(p) \right]$

- If the impedance is constant (in the frequency range of interest)

$$\omega_s^2 = \omega_{s0}^2 \left\{ 1 - \frac{2 \, \pi \, I_b}{\hat{V}_{\mathrm{RF}} \, h \cos \phi_{s0}} \left[j \, \frac{Z_l(p)}{p} \right]_{const} \sum_{p=-\infty}^{p=+\infty} p^2 \, \sigma_0(p) \right\}$$

EFFECT OF THE STATIONARY DISTRIBUTION (6/9)

- **Using the relation** $\boxed{\sum_{p=-\infty}^{p=+\infty} J_2(p\,x) = \frac{2}{x}}$ $\Rightarrow \sum_{p=-\infty}^{p=+\infty} p^2\,\sigma_0(p) = \frac{8}{\pi^4\,B^3}$

For the parabolic amplitude density

$$\Rightarrow \quad \boxed{\Delta = \frac{\omega_s^2 - \omega_{s0}^2}{\omega_{s0}^2} = -\frac{16\,I_b}{\pi^3\,B^3\,\hat{V}_{\mathrm{RF}}\,h\cos\phi_{s0}}\left[j\,\frac{Z_l(p)}{p}\right]_{const}}$$

The change in the RF slope corresponds to the **effective (total) voltage**

$$\hat{V}_{\mathrm{T}} = \hat{V}_{\mathrm{RF}}\left(\frac{\omega_s}{\omega_{s0}}\right)^2$$

EFFECT OF THE STATIONARY DISTRIBUTION (7/9)

- **Bunch lengthening / shortening** (as a consequence of the shifts of the synchronous phase and incoherent frequency)

  - **Electrons**

The equilibrium momentum spread is imposed by synchrotron radiation

$$\boxed{\frac{\Delta p}{p_0} = \left(\frac{\Delta p}{p_0}\right)_0} \quad \Rightarrow \quad \frac{B}{B_0} = \frac{\omega_{s0}}{\omega_s}\sqrt{\left|\frac{\cos\phi_{s0}}{\cos\phi_s}\right|}$$

Neglecting the (usually small) synchronous phase shift

$$\Rightarrow \quad \frac{B}{B_0} = \left(\frac{B}{B_0}\right)^3 + \Delta_0 \qquad \text{with} \qquad \boxed{\Delta_0 = \Delta_{B = B_0}}$$

EFFECT OF THE STATIONARY DISTRIBUTION (8/9)

- **Protons**

The longitudinal emittance is invariant

$$\boxed{\tau_b \frac{\Delta p}{p_0} = \tau_{b0} \left(\frac{\Delta p}{p_0} \right)_0} \quad \Rightarrow \quad \left(\frac{B}{B_0} \right)^2 = \frac{\omega_{s0}}{\omega_s} \sqrt{\left| \frac{\cos \phi_{s0}}{\cos \phi_s} \right|}$$

Again, neglecting the (usually small) synchronous phase shift

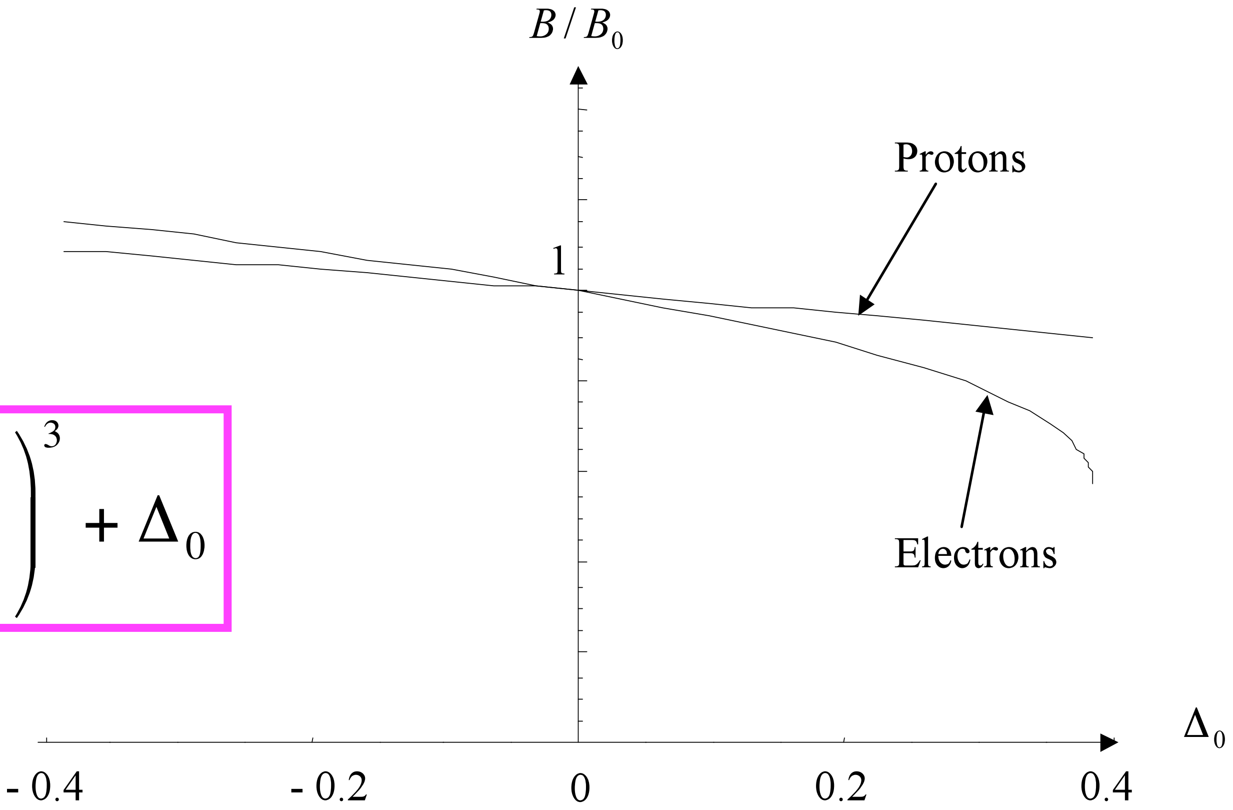
$$\Rightarrow \quad \left(\frac{B}{B_0} \right)^{-1} = \left(\frac{B}{B_0} \right)^3 + \Delta_0$$

EFFECT OF THE STATIONARY DISTRIBUTION (9/9)

- **General formula**

+ for electrons and – for protons

$$\boxed{\left(\frac{B}{B_0}\right)^{\pm 1} = \left(\frac{B}{B_0}\right)^{3} + \Delta_0}$$



◆ Conclusion of the effect of the stationary distribution: **New fixed point**

- Synchronous phase shift $\phi_{s0} \Rightarrow \phi_s \left(I_b \right)$
- Potential-well distortion

$$\left.\begin{array}{l} \phi_{s0} \Rightarrow \phi_s \left(I_b \right) \\ \hat{V}_{\mathrm{RF}} \Rightarrow \hat{V}_{\mathrm{T}} \left(I_b \right) \quad \omega_{s0} \Rightarrow \omega_s \left(I_b \right) \end{array}\right\} \quad B_0 \Rightarrow B \left(I_b \right)$$

PERTURBATION DISTRIBUTION (1/2)

Around the new fixed point

- The form is suggested by the single-particle signal

$$s_z(t,\vartheta) = \frac{e\,\Omega_0}{2\pi} \sum_{p,m=-\infty}^{p,m=+\infty} j^{-m} J_m(p\,\Omega_0\,\hat{\tau})\, e^{j(\omega_{pm} t - p\vartheta + m\psi_0)}$$

- **Low-intensity** $\boxed{\Delta\Psi(\hat{\tau},\psi_0,t) = g_m(\hat{\tau})\, e^{-jm\psi_0}\; e^{j\Delta\omega_{cm} t}}$ $\quad m \neq 0$

$$\Delta\omega_{cm} = \omega_c - m\,\omega_s << \omega_{s0}$$

Coherent synchrotron frequency shift to be determined

Therefore, the spectral amplitude is maximum for satellite number *m* and null for the other satellites

PERTURBATION DISTRIBUTION (2/2)

$$\Rightarrow \quad \boxed{\Delta S_{zm}(\omega,\vartheta) = 2\pi I_b \sum_{p=-\infty}^{p=+\infty} \sigma_m(p)\,\delta\left[\,\omega - (p\,\Omega_0 + m\,\omega_s + \Delta\omega_{cm})\right] e^{-j\,p\,\vartheta}}$$

with $$\boxed{\sigma_m(p) = j^{-m} \int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} J_m(p\,\Omega_0\,\hat{\tau})\, g_m(\hat{\tau})\,\hat{\tau}\,d\hat{\tau}}$$

Amplitude of the perturbation spectrum

$$\omega_s = \Omega_0 \left(-\frac{e\,\hat{V}_{\mathrm{T}}\,h\,\eta\cos\phi_s}{2\pi\,\beta^2\,E_{total}} \right)^{1/2}$$

- **High-intensity** $$\boxed{\Delta\Psi(\hat{\tau},\psi_0,t) = \sum_m g_m(\hat{\tau})\,e^{-j\,m\,\psi_0}\,e^{\,j\,\Delta\omega_{cm}\,t}}$$

EFFECT OF THE PERTURBATION (1/10)

$$\Psi(\hat{\tau}, \psi_0, t) = \Psi_0 + \Delta\Psi = g_0(\hat{\tau}) + \sum_m g_m(\hat{\tau})\, e^{-j m \psi_0}\, e^{j \Delta\omega_{cm} t}$$

- Vlasov equation with variables $(\hat{\tau}, \psi_0)$

$$\frac{\partial \Psi}{\partial t} + \left(\frac{d g_0}{d \hat{\tau}} + \frac{\partial \Delta\Psi}{\partial \hat{\tau}} \right) \frac{d \hat{\tau}}{dt} + \frac{\partial \Delta\Psi}{\partial \psi_0} \frac{d \psi_0}{dt} = 0$$

$\Rightarrow$ **Linearized Vlasov equation** $\boxed{\dfrac{\partial \Psi}{\partial t} = - \dfrac{d g_0}{d \hat{\tau}} \dfrac{d \hat{\tau}}{dt}}$

$$\Rightarrow \quad j \sum_m g_m(\hat{\tau})\, e^{-j m \psi_0}\, \Delta\omega_{cm}\, e^{j \Delta\omega_{cm} t} = - \frac{d g_0}{d \hat{\tau}} \frac{d \hat{\tau}}{dt}$$

EFFECT OF THE PERTURBATION (2/10)

$$\boxed{\frac{d\hat{\tau}}{dt} = \frac{d}{dt}\left(\sqrt{\tau^2 + \frac{\dot{\tau}^2}{\omega_s^2}}\right) = -\frac{F_c}{\omega_s}\sin\left(\omega_s t + \psi_0\right)}$$

with $\quad \ddot{\tau} + \omega_s^2\,\tau = F_c = \dfrac{\eta\, e}{p_0}\left[\vec{E} + \vec{v} \times \vec{B}\right]_{zc}\left(t, \vartheta = \Omega_0\left(t - \tau\right)\right)$

$$\Rightarrow \quad \boxed{F_c = \frac{2\pi\, I_b\, \omega_s^2}{\Omega_0\, \hat{V}_{\mathrm{T}}\, h \cos\phi_s}\, e^{\,j\,\omega_c\, t} \sum_{p=-\infty}^{p=+\infty} Z_l\left(p\right) e^{\,j\,p\,\Omega_0\,\tau}\,\sigma\left(p\right)}$$

with $\quad \boxed{\sigma\left(p\right) = \sum_m \sigma_m\left(p\right)}$

Spectrum amplitude at frequency $p\,\Omega_0 + \omega_c$

EFFECT OF THE PERTURBATION (3/10)

- Expanding the product $\sin\psi \, e^{\,j\,p\,\Omega_0\,\tau}$ (using previously given relations)

$$\psi = \omega_s \, t + \psi_0$$

$$\boxed{\sin\psi \, e^{\,j\,p\,\Omega_0\,\tau} = \sum_{m=-\infty}^{m=+\infty} j^{\,m} \, e^{-j\,m\,\psi} \, \frac{m}{p\,\Omega_0\,\hat{\tau}} \, J_m\left(p\,\Omega_0\,\hat{\tau}\right)}$$

$\Longrightarrow$ **Final form of the equation of coherent motion of a single bunch:**

$$\Delta\omega_{cm} = \omega_c - m\,\omega_s$$

Contribution from all the modes *m*

$$\boxed{j\,\Delta\omega_{cm}\, j^{-m}\, g_m\left(\hat{\tau}\right)\hat{\tau} = \frac{2\pi\, I_b\, m\,\omega_s}{\Omega_0^2\,\hat{V}_{\mathrm{T}}\, h\cos\phi_s}\,\frac{d g_0}{d\hat{\tau}} \sum_{p=-\infty}^{p=+\infty} \frac{Z_l\left(p\right)}{p}\, J_m\left(p\,\Omega_0\,\hat{\tau}\right)\sigma\left(p\right)}$$

EFFECT OF THE PERTURBATION (4/10)

◆ Coherent modes of oscillation **at low intensity** (i.e. considering only a single mode *m*)

$$j\,\Delta\omega_{cm}\;j^{-m}\;g_m(\hat{\tau})\,\hat{\tau} = \frac{2\pi I_b\,m\,\omega_s}{\Omega_0^2\,\hat{V}_T\;h\cos\phi_s}\,\frac{dg_0}{d\hat{\tau}}\sum_{p=-\infty}^{p=+\infty}\frac{Z_l(p)}{p}\,J_m(p\,\Omega_0\,\hat{\tau})\,\sigma_m(p)$$

Multiplying both sides by $J_m(l\,\Omega_0\,\hat{\tau})$ and integrating over $\hat{\tau}$

$$\Rightarrow \boxed{\Delta\omega_{cmq}\;\sigma_{mq}(l) = \sum_{p=-\infty}^{p=+\infty} K_{lp}^m\,\sigma_{mq}(p)}$$

Twofold infinity of coherent modes

$$\boxed{\Delta\omega_{cmq} = \omega_{cmq} - m\,\omega_s}$$

$$\boxed{K_{lp}^m = -\frac{2\pi I_b\,m\,\omega_s}{\Omega_0^2\,\hat{V}_T\;h\cos\phi_s}\;j\,\frac{Z_l(p)}{p}\int_{\hat{\tau}=0}^{\hat{\tau}=+\infty}\frac{dg_0}{d\hat{\tau}}\,J_m(p\,\Omega_0\,\hat{\tau})\,J_m(l\,\Omega_0\,\hat{\tau})\,d\hat{\tau}}$$

EFFECT OF THE PERTURBATION (5/10)

- The procedure to obtain first order exact solutions, with realistic modes and a general interaction, thus consists of finding the eigenvalues and eigenvectors of the infinite complex matrix whose elements are K_{lp}^{m}

- The result is an infinite number of modes $m\,q$ ($-\infty < m\,, q < +\infty$) of oscillation (as there are 2 degrees of freedom $\left(\hat{\tau}\,, \psi_0 \right)$)

The imaginary part tells us if this mode is stable or not

- To each mode, one can associate:
  - a coherent frequency shift $\Delta\omega_{cmq} = \omega_{cmq} - m\,\omega_s$ (**qth eigenvalue**)
  - a coherent spectrum $\sigma_{mq}\left(p \right)$ (**qth eigenvector**)
  - a perturbation distribution $g_{mq}\left(\hat{\tau} \right)$

- For numerical reasons, the matrix needs to be truncated, and thus only a finite frequency domain is explored

EFFECT OF THE PERTURBATION (6/10)

- The longitudinal signal at the PU electrode is given by

$$\boxed{S_{mq}(t,\vartheta) = S_{z0}(t,\vartheta) + \Delta S_{zmq}(t,\vartheta)}$$

$$S_{z0}(t,\vartheta) = 2\pi I_b \sum_{p=-\infty}^{p=+\infty} \sigma_0(p)\, e^{j p \Omega_0 t}\, e^{-j p \vartheta}$$

$$\Delta S_{zmq}(t,\vartheta) = 2\pi I_b \sum_{p=-\infty}^{p=+\infty} \sigma_{mq}(p)\, e^{j\left(p\Omega_0 + m\,\omega_s + \Delta\omega_{cmq}\right)t}\, e^{-j p \vartheta}$$

- For the case of the parabolic amplitude distribution

$$g_0(\hat{z}) = \frac{2}{\pi\left(\dfrac{\tau_b}{2}\right)^2}\left(1-\hat{z}^2\right) \qquad S_{z0}(t,\vartheta) = 8\, I_b \sum_{p=-\infty}^{p=+\infty} e^{j p \Omega_0 t}\, e^{-j p \vartheta}\, \frac{J_2(p\,\pi\,B)}{(p\,\pi\,B)^2}$$

EFFECT OF THE PERTURBATION (7/10)

$$K_{lp}^{m} = \frac{128\, I_b\, m\, \omega_s}{\Omega_0^2\, \hat{V}_{\mathrm{T}}\, h \cos\phi_s\, \tau_b^4}\, j\, \frac{Z_l(p)}{p} \int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} J_m(p\,\Omega_0\,\hat{\tau})\, J_m(l\,\Omega_0\,\hat{\tau})\,\hat{\tau}\, d\hat{\tau}$$

- Low order eigenvalues and eigenvectors of the matrix can be found quickly by computation, **using the relations**

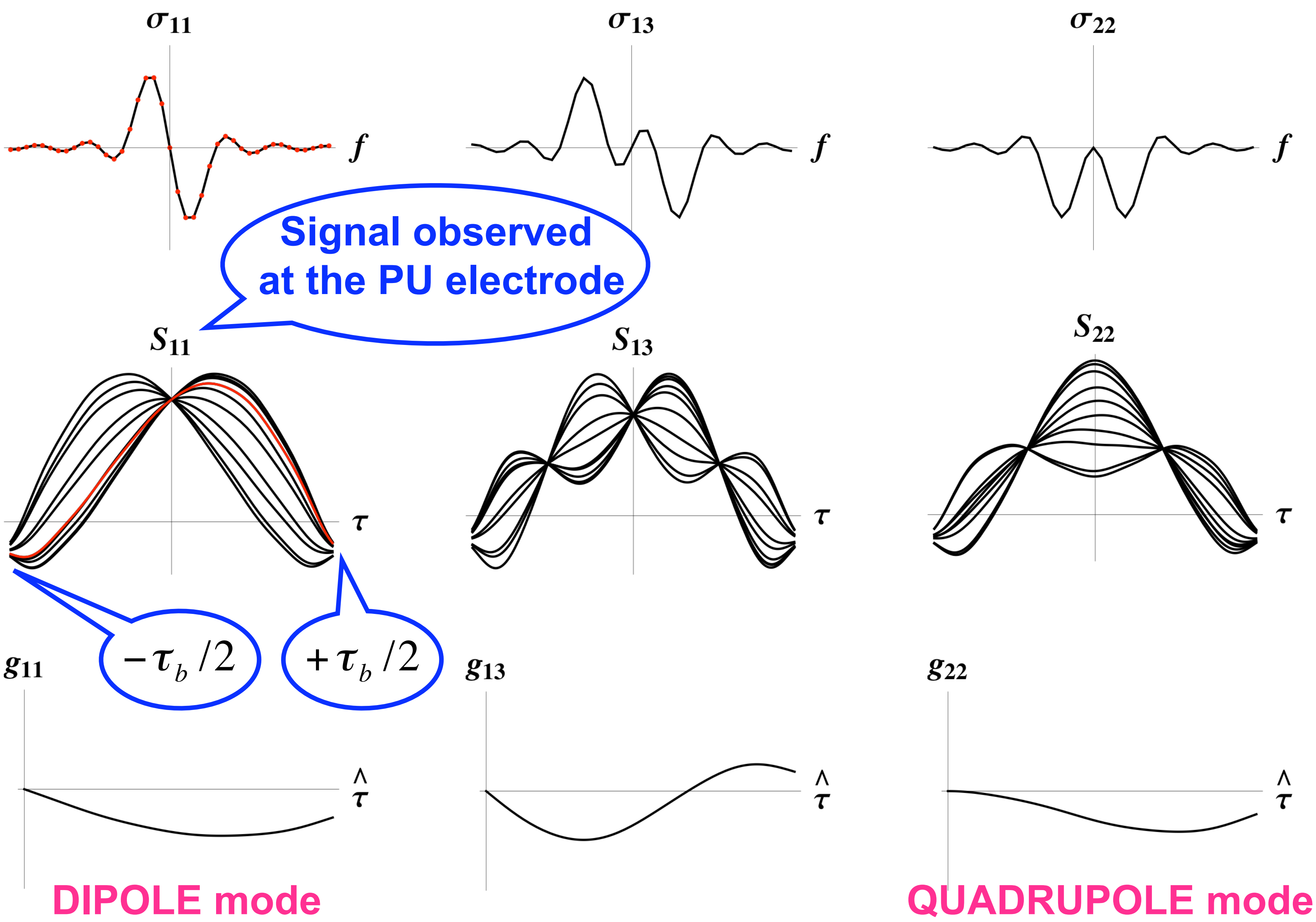
$$\boxed{\int_0^X J_m^2(a\,x)\,x\,dx = \frac{X^2}{2}\left[J_m'(a\,X)\right]^2 + \frac{1}{2}\left[X^2 - \frac{m^2}{a^2}\right] J_m^2(a\,X)}$$

$$\boxed{\int_0^X x\,J_m(a\,x)\,J_m(b\,x)\,dx = \frac{X}{a^2 - b^2}\left[a\,J_m(b\,X)\,J_{m+1}(a\,X) - b\,J_m(a\,X)\,J_{m+1}(b\,X)\right]}$$

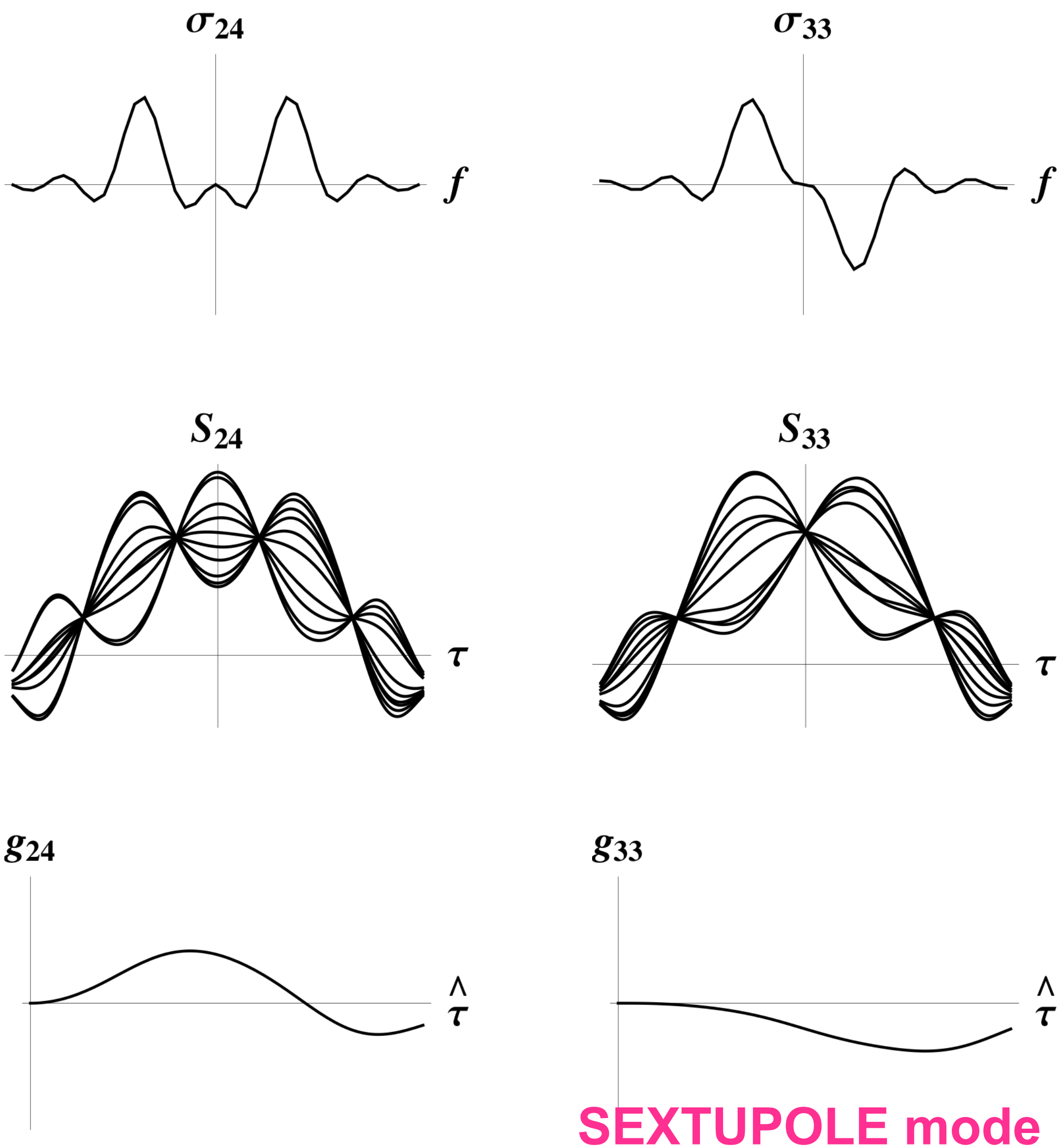
$$\boxed{a^2 \neq b^2}$$

- The case of a constant inductive impedance is solved in the next slides, and the signal at the PU shown for several superimposed turns

EFFECT OF THE PERTURBATION (8/10)



EFFECT OF THE PERTURBATION (9/10)



SEXTUPOLE mode

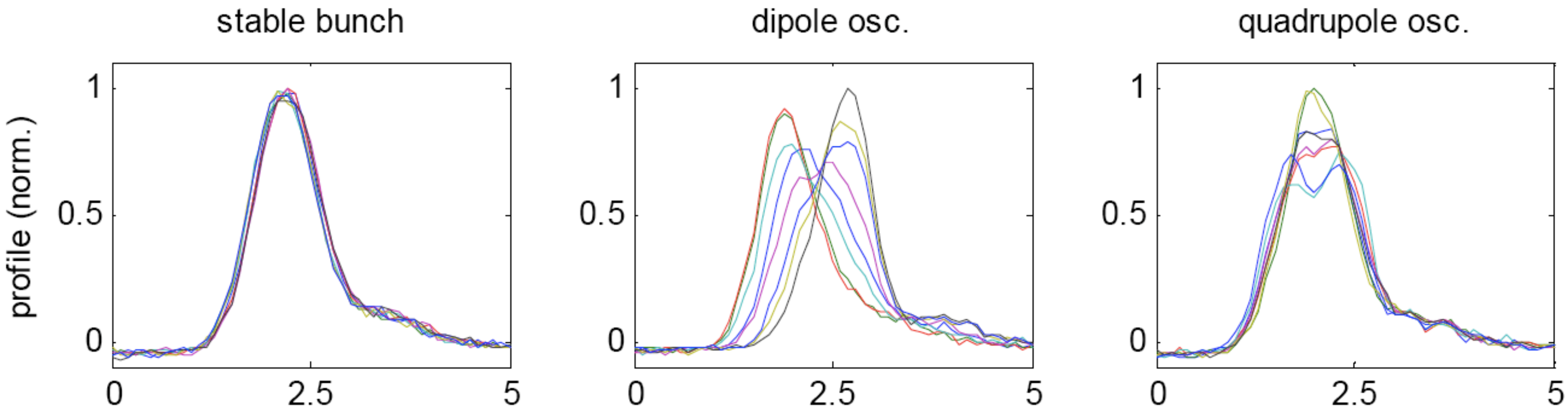
- **The spectrum of mode *mq* is peaked at** $\boxed{f_q \approx \frac{q+1}{2\,\tau_b}}$ **and extends** $\boxed{\sim \pm\, \tau_b^{-1}}$

  $\boxed{q \equiv m + 2\,k}$ $\quad 0 \leq k < +\infty$

- There are ***q*** **nodes** on these "**standing-wave**" patterns

EFFECT OF THE PERTURBATION (10/10)

Observations in the CERN SPS in 2007



(Laclare's) theory