TRANSVERSE BEAM DYNAMICS

Elias Métral

- Introduction
- Multipole field expansion
- Equations of motion (general & Hill’s equation and solution)
- Matrix formalism and Twiss parameters
- Thin lens approximation and FODO cell
- Beam emittance, envelope and divergence
- Dispersion
- Normalized Floquet’s coordinates
- Chromaticity and its correction
- Nonlinearities and resonances
The motion of a charged particle (proton) in a beam transport channel or a circular accelerator is governed by the **Lorentz force**

\[ \vec{F} = \frac{d\vec{p}}{dt} = e \left( \vec{E} + \vec{v} \times \vec{B} \right) \]

The motion of particle beams under the influence of the Lorentz force is called **beam optics**.
The Lorentz force is applied as a

- **BENDING FORCE** (using DIPOLES) to guide the particles along a predefined ideal path, the DESIGN ORBIT, on which – ideally – all particles should move

- **FOCUSING FORCE** (using QUADRUPOLES) to confine the particles in the vicinity of the ideal path, from which most particles will unavoidably deviate

**LATTICE** = Arrangement of magnets along the design orbit

**The ACCELERATOR DESIGN** is made considering the beam as a collection of non-interacting single particles
⇒ A particle, with a constant energy, describes a circle in equilibrium between the centripetal magnetic force and the centrifugal force.

\[ e v B = m \frac{v^2}{\rho} \Rightarrow B \rho = \frac{p}{e} \]

- **BEAM RIGIDITY** (in useful units)

\[ B \rho [\text{Tm}] = 3.3356 p_0 [\text{GeV/c}] \]

**Magnetic field**

**Curvature radius of the dipoles**

**Beam momentum**
INTRODUCTION (4/6)

- LEP vs LHC magnets (in same tunnel) => A change in technology

<table>
<thead>
<tr>
<th></th>
<th>LEP</th>
<th>LHC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$ [m]</td>
<td>3096.175</td>
<td>2803.95</td>
</tr>
<tr>
<td>$p_0$ [GeV/c]</td>
<td>104</td>
<td>7000</td>
</tr>
<tr>
<td>$B$ [T]</td>
<td>0.11</td>
<td>8.33</td>
</tr>
</tbody>
</table>

- Room-temperature coils
- Superconducting coils

- As the machine has to be closed

$$2\pi = N_d \vartheta_d$$

$$l_d = \rho \vartheta_d$$
... and SEXTUPOLES (6 poles), OCTUPOLES (8 poles), etc.

In $x$ (and Defocusing in $y$) $\Rightarrow$ F-type. Permutating the N- and S- poles gives a D-type.
(fast) EXTRACTION from a ring

SEPTUM
(field ≠ 0 between the 2 blades)

KICKER
(fast dipole < 1 turn)

(fast) INJECTION into a ring => Reverse process
In a vacuum environment in the vicinity of the design orbit, the following Maxwell equations hold:

\[ \nabla \cdot \vec{B} = 0 \]
\[ \nabla \times \vec{B} = 0 \]

Assumption that the field does not vary along the z-axis (as is the case for long magnets far from the ends) and that there are only transverse field components (no solenoid field):

\[ \nabla \times \vec{B} = 0 \implies \vec{B} = -\nabla U \]

\[ \Delta U \equiv \nabla^2 U = 0 \]

With \( U \) a magnetic scalar potential.

Laplace equation.
The real and imaginary parts are 2 independent solutions of the Laplace equation and they differentiate between 2 classes of magnet orientation.
The imaginary part has the so-called mid-plane symmetry, i.e.

\[ \text{Im} \left[ U_n(x, y) \right] = - \text{Im} \left[ U_n(x, -y) \right] \]

and the magnetic field components for the nth order multipoles are

\[
B_{nx}(x, y) = -\frac{\partial}{\partial x} \text{Im} \left[ U_n(x, y) \right] = \frac{p}{e} A_n \left( \frac{n-2}{2} \right) \sum_{m=0}^{(n-2)/2} (-1)^m \frac{x^{n-2m-2} y^{2m+1}}{(n-2m-2)! (2m+1)!}
\]

\[
B_{ny}(x, y) = -\frac{\partial}{\partial y} \text{Im} \left[ U_n(x, y) \right] = \frac{p}{e} A_n \left( \frac{n-1}{2} \right) \sum_{m=0}^{(n-1)/2} (-1)^m \frac{x^{n-2m-1} y^{2m}}{(n-2m-1)! (2m)!}
\]
The mid-plane symmetry yields:

\[
B_{nx}(x, y) = -B_{nx}(x, -y) \\
B_{ny}(x, y) = B_{ny}(x, -y)
\]

There is no horizontal field in the mid-plane, \(B_{nx}(x, 0) = 0\), and a particle traveling in the horizontal mid-plane will remain in this plane.

The magnets are called UPRIGHT or REGULAR magnets.

The coefficient \(A_n\) is called the (normalized) multipole strength parameter:

\[
A_n = \frac{e}{p} \frac{\partial^{n-1}}{\partial x^{n-1}} B_{ny} = -\frac{e}{p} (-1)^{n/2} \frac{\partial^{n-1}}{\partial y^{n-1}} B_{nx}
\]
MULTIPOLE FIELD EXPANSION (5/7)

- The magnets derived from the real solution of the potential are called **ROTATED** or **SKEW** magnets and the magnetic field components for the nth order skew multipoles are

\[
B_{nx}(x, y) = -\frac{\partial}{\partial x} \text{Re} \left[ U_n(x, y) \right]
\]

\[
= \frac{p}{e} A_n \sum_{m=0}^{(n-1)/2} (-1)^m \frac{x^{n-2m-1}}{(n-2m-1)!} \frac{y^{2m}}{(2m)!}
\]

\[
B_{ny}(x, y) = -\frac{\partial}{\partial y} \text{Re} \left[ U_n(x, y) \right]
\]

\[
= \frac{p}{e} A_n \sum_{m=0}^{n/2} (-1)^m \frac{x^{n-2m}}{(n-2m)!} \frac{y^{2m-1}}{(2m-1)!}
\]
MULTIPOLE FIELD EXPANSION (6/7)

• The coefficient $A_n$ is called the (normalized) skew multipole strength parameter

$$A_n = \frac{e}{p} \frac{\partial^{n-1}}{\partial x^{n-1}} B_{nx} = \frac{e}{p} \left(-1\right)^{n/2} \frac{\partial^{n-1}}{\partial y^{n-1}} B_{ny}$$

• The skew magnets differ from the regular ones only by a rotation about the z-axis by an angle (where $n$ is the order of the multipole)

$$\phi_n = \frac{\pi}{2n}$$
### Regular multipole fields

<table>
<thead>
<tr>
<th>Multipole</th>
<th>Expression</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Dipole</strong></td>
<td>$\frac{e}{p} B_{1x} = 0$</td>
<td>$\frac{e}{p} B_{1y} = \frac{1}{\rho_x}$</td>
</tr>
<tr>
<td><strong>Quadrupole</strong></td>
<td>$\frac{e}{p} B_{2x} = K_y$</td>
<td>$\frac{e}{p} B_{2y} = K_x$</td>
</tr>
<tr>
<td><strong>Sextupole</strong></td>
<td>$\frac{e}{p} B_{3x} = S_{xy}$</td>
<td>$\frac{e}{p} = B_{3y} = \frac{1}{2} S(x^2 - y^2)$</td>
</tr>
<tr>
<td><strong>Octupole</strong></td>
<td>$\frac{e}{p} B_{4x} = \frac{1}{6} O(3x^2y - y^3)$</td>
<td>$\frac{e}{p} B_{4y} = \frac{1}{6} O(x^3 - 3xy^2)$</td>
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### Skew multipole fields

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</thead>
<tbody>
<tr>
<td><strong>Dipole</strong></td>
<td>$\frac{e}{p} B_{1x} = -\frac{1}{\rho_y}$</td>
<td>$\frac{e}{p} B_{1y} = 0$</td>
</tr>
<tr>
<td>($\phi = 90^\circ$)</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Quadrupole</strong></td>
<td>$\frac{e}{p} B_{2x} = K_x$</td>
<td>$\frac{e}{p} B_{2y} = -K_y$</td>
</tr>
<tr>
<td>($\phi = 45^\circ$)</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Sextupole</strong></td>
<td>$\frac{e}{p} B_{3x} = \frac{1}{2} S(x^2 - y^2)$</td>
<td>$\frac{e}{p} B_{3y} = -S_{xy}$</td>
</tr>
<tr>
<td>($\phi = 30^\circ$)</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Octupole</strong></td>
<td>$\frac{e}{p} B_{4x} = \frac{1}{6} O(x^3 - 3xy^2)$</td>
<td>$\frac{e}{p} B_{4y} = -\frac{1}{6} O(3x^2y - y^3)$</td>
</tr>
<tr>
<td>($\phi = 22.5^\circ$)</td>
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</table>
EQUATIONS OF MOTION (1/9)

**General equations of motion for a charged particle in a magnetic field** (in absence of electric field)

\[
x'' - \frac{\sigma''}{\sigma'} x' = k_x \ h - \left(1 + \delta\right)^{-1} \frac{e}{p_0} \sigma' \left(h B_y - y' B_t\right)
\]

\[
y'' - \frac{\sigma''}{\sigma'} y' = k_y \ h + \left(1 + \delta\right)^{-1} \frac{e}{p_0} \sigma' \left(h B_x - x' B_t\right)
\]

with

\[
\delta = \frac{p - p_0}{p_0} = \frac{\Delta p}{p_0}
\]

\[
h = 1 + k_x \ x + k_y \ y
\]

\[
\sigma' = \sqrt{h^2 + x'^2 + y'^2}
\]

\[
\rho_{x,y} (s) = \frac{1}{k_{x,y}(s)}
\]

Particle trajectory

On the design orbit, \(x = x' = y = y' = \delta = 0\)
General equations of motion expanded to the 2\textsuperscript{nd} order in $x$, $y$ and $\delta$ (without tangential fields)

\begin{align*}
x'' + \left( K_0 + \frac{1}{\rho_x^2} \right) x &= K_0 \ y + \frac{\delta}{\rho_x} - \frac{\delta^2}{\rho_x} + \left( K_0 + \frac{2}{\rho_x^2} \right) x \ \delta - \frac{1}{2} \ S_0 \left( x^2 - y^2 \right) - K_0 \ y \ \delta + S_0 \ x \ y \\
y'' - \left( K_0 - \frac{1}{\rho_y^2} \right) y &= K_0 \ x + \frac{\delta}{\rho_y} - \frac{\delta^2}{\rho_y} - \left( K_0 - \frac{2}{\rho_y^2} \right) y \ \delta + \frac{1}{2} \ S_0 \left( x^2 - y^2 \right) - K_0 \ x \ \delta + S_0 \ x \ y
\end{align*}

If we consider the on-momentum particle ($\delta = 0$) and if we assume that the design orbit lies only in the horizontal plane, then the equations of motion simplifies to

\begin{align*}
x'' + \left( K_0 + \frac{1}{\rho_x^2} \right) x &= K_0 \ y - \frac{1}{2} \ S_0 \left( x^2 - y^2 \right) + S_0 \ x \ y \\
y'' - K_0 \ y &= K_0 \ x + \frac{1}{2} \ S_0 \left( x^2 - y^2 \right) + S_0 \ x \ y
\end{align*}
EQUATIONS OF MOTION (3/9)

- If we consider now the on-momentum particle \((\delta = 0)\), if we assume that the design orbit lies only in the horizontal plane, and if we consider only the presence of dipoles and normal (regular) quadrupoles, then the equations of motion simplifies to the so-called HILL’S EQUATION

\[
x'' + \left( K_0 + \frac{1}{\rho_x^2} \right) x = 0
\]

\[
y'' - K_0 y = 0
\]

Periodic function of the \(s\)-coordinate with period \(L = C / N_{\text{cell}}\)

where \(C\) is the accelerator circumference and \(N_{\text{cell}}\) the number of “cells” repeated

\[
u'' + K(s) u = 0
\]

\(u = x\) or \(y\)

Gradient focusing

Weak sector magnet focusing
Reminder on light optics

Principle of focusing for light

\[
\tan \theta = -\frac{x}{f}
\]

Principle of **STRONG FOCUSING** for light

\[
f = \frac{f_F^2}{d} > 0 \quad \text{(when } f_D = -f_F \text{)}
\]

=> Focusing in both planes
Coming back to the case of a strong focusing lattice

\[ \int_0^l u'' \, ds = u'(l) - u'(0) = \tan \theta \]

\[ \int_0^l K(s) \, u \, ds \approx K \, u \, l \]

\[ \tan \theta \approx - K \, u \, l \]

By analogy with the expression of the focal length of a glass lens, we define the focal length of a quadrupole

\[ \tan \theta = - \frac{u}{f} \]

\[ \frac{1}{f} = \pm K_0 \, l \]

Assuming \( K(s) \approx K_0 > 0 \)

+ in focusing plane and
– in defocusing plane
The solution of the Hill’s equation is a pseudo-harmonic oscillation with varying amplitude and frequency called BETATRON OSCILLATION

\[ u(s) = a \sqrt{\beta(s)} \cos[\mu(s) - \varphi] \]

with

\[ \mu(s) = \int_{s_0}^{s} \frac{dt}{\beta(t)} \]

\[ \frac{1}{2} \beta \beta'' - \frac{1}{4} \beta'^2 + K(s) \beta^2 = 1 \]

Phase advance per period or cell (of length \( L \))

\[ \mu = \int_{s}^{s+L} \frac{dt}{\beta(t)} \]
EQUATIONS OF MOTION (7/9)

- The oscillation’s local frequency $f(s)$ is related to the phase function by

$$\mu'(s) = 2\pi f(s)$$

- The number of betatron oscillations executed by particles traveling once around the machine circumference $C$ is called the BETATRON TUNE and is given by

$$Q = \int_{s}^{s+C} f(t) \, dt$$

$$=> Q = \frac{N_{cell} \mu}{2\pi} = \frac{1}{2\pi} \int_{s}^{s+C} \frac{dt}{\beta(t)}$$
EQUATIONS OF MOTION (8/9)

- Case of a drift space (where $K = 0$)

$$\frac{1}{2} \beta \beta'' - \frac{1}{4} \beta'^2 + K(s) \beta^2 = 1$$

$$\Rightarrow \frac{1}{2} \beta \beta'' - \frac{1}{4} \beta'^2 = 1$$

$$\Rightarrow$$ The solution is (with $\beta(0) = \beta_0$)

$$\beta(s) = \beta_0 + \beta_0' s + \frac{1}{\beta_0} \left[1 + \frac{\beta_0'^2}{4}\right] s^2$$

- When $\beta'_0 = 0$, the betatron function reduces to

Case of a symmetric point, as a LHC IP
The betatron phase advance is given by

\[ \mu(s) = \int_0^s \frac{dt}{\beta(t)} = \arctan \left( \frac{s}{\beta^*} \right) \]
Using the matrix formalism, one can write

\[
\begin{bmatrix}
u(s) \\
\nu'(s)
\end{bmatrix} = \begin{bmatrix} M(s / s_0) & 0 \\
0 & M(s / s_0)
\end{bmatrix}
\begin{bmatrix}
u(s_0) \\
\nu'(s_0)
\end{bmatrix}
\]

with

\[
M(s / s_0) = \begin{bmatrix}
\sqrt{\frac{\beta(s)}{\beta_0}} \left[ \cos \Delta \mu(s) + \alpha_0 \sin \Delta \mu(s) \right] & \sqrt{\beta(s) \beta_0} \sin \Delta \mu(s) \\
\sqrt{\frac{\alpha_0 - \alpha(s)}{\sqrt{\beta(s)} \beta_0}} \cos \Delta \mu(s) - \frac{1 + \alpha(s) \alpha_0}{\sqrt{\beta(s)} \beta_0} \sin \Delta \mu(s) & \sqrt{\frac{\beta_0}{\sqrt{\beta(s)}}} \left[ \cos \Delta \mu(s) - \alpha(s) \sin \Delta \mu(s) \right]
\end{bmatrix}
\]

\[
\alpha(s) = -\frac{\beta'(s)}{2} \\
\Delta \mu(s) = \mu(s) - \mu(s_0)
\]
Effect of the kick of a dipole => Consider a small dipole placed at $s_0$ yielding an angular kick

$$\vartheta = \frac{l}{\rho} = \frac{e}{p} \Delta B \, l$$

The motion of a particle traveling on the design orbit for $s > s_0$ (before it is 0) is (from the previous general matrix formula)

$$x(s) = \sqrt{\beta(s)} \beta_0 \sin \left[ \mu(s) - \mu_0 \right] \vartheta$$

$$\beta(s_0) = \beta_0$$

$$\mu(s_0) = \mu_0$$
The transfer matrix over one period = **TWISS MATRIX** and is given by

\[
M(s) \equiv M(s + L/s) = \begin{bmatrix}
\cos \mu + \alpha(s) \sin \mu & \beta(s) \sin \mu \\
-\gamma(s) \sin \mu & \cos \mu - \alpha(s) \sin \mu
\end{bmatrix}
\]

with

\[
\gamma(s) = \frac{1 + \alpha(s)^2}{\beta(s)}
\]

- **Determinant of the Twiss matrix**
  \[
  \det[M(s)] = 1
  \]

- **Trace of the Twiss matrix**
  \[
  \text{Tr}[M(s)] = 2 \cos \mu
  \]

\(\beta(s), \alpha(s)\) and \(\gamma(s)\) are called **TWISS PARAMETERS**

**General stability criterion** for a Twiss matrix

\[|\text{Tr}[M(s)]| \leq 2\]
Hill’s equation with piecewise periodic constant coefficients =>
Assume $K(s)$ to be a constant $K$ over a distance $l$ between the azimuthal locations $s_0$ and $s_1$: $> 0$, $< 0$ or $= 0$

- **For $K > 0$**
  
  $$M\left(\frac{s_1}{s_0}\right) = \begin{bmatrix} \cos\left(\sqrt{K} \ l\right) & \frac{1}{\sqrt{K}} \sin\left(\sqrt{K} \ l\right) \\ -\sqrt{K} \sin\left(\sqrt{K} \ l\right) & \cos\left(\sqrt{K} \ l\right) \end{bmatrix}$$

- **For $K < 0$**
  
  $$M\left(\frac{s_1}{s_0}\right) = \begin{bmatrix} \cosh\left(\sqrt{|K|} \ l\right) & \frac{1}{\sqrt{|K|}} \sinh\left(\sqrt{|K|} \ l\right) \\ \sqrt{|K|} \sinh\left(\sqrt{|K|} \ l\right) & \cosh\left(\sqrt{|K|} \ l\right) \end{bmatrix}$$

- **For $K = 0$ (drift space)**
  
  $$M\left(\frac{s_1}{s_0}\right) = \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix}$$
If the cell is made of $N$ elements having transfer matrices $M_1, M_2, \ldots, M_N$

$$M_k = M \left( \frac{s_k}{s_{k-1}} \right)$$

$$\Rightarrow M \left( s_0 \right) \equiv M \left( s_0 + \frac{L}{s_0} \right) = M_N M_{N-1} \cdots M_2 M_1$$

Noting $m_{ij} \left( s_0 \right)$ the components of the matrix $M \left( s_0 \right)$, the Twiss parameters at the reference point $s_0$ are given by

$$\beta \left( s_0 \right) = \frac{m_{12} \left( s_0 \right)}{\sin \mu}$$

$$\alpha \left( s_0 \right) = \frac{m_{11} \left( s_0 \right) - m_{22} \left( s_0 \right)}{2 \sin \mu}$$

$$\gamma \left( s_0 \right) = - \frac{m_{21} \left( s_0 \right)}{\sin \mu}$$

$$\cos \mu = \frac{1}{2} \left[ m_{11} \left( s_0 \right) + m_{22} \left( s_0 \right) \right]$$
Thin lens approximation

\[ \sqrt{K} l \ll 1 \quad \text{with} \quad K > 0 \]

\[ l \to 0 \quad \text{with} \quad K l = \text{constant} \]

=> Transfer matrix of a thin quadrupole is

\[
M \left( \frac{l}{0} \right) = \begin{bmatrix}
1 & 0 \\
\mp K l & 1
\end{bmatrix}
\]

- => Horizontally focusing quadrupole
+ => Horizontally defocusing quadrupole
THIN LENS APPROXIMATION AND FODO CELL (2/4)

- Transfer matrix of a symmetric thin-lens FODO cell

\[
M_{\text{FODO}} \left( 2 \, L / 0 \right) = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ f^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -f^{-1} & 1 \end{pmatrix} \\
= \begin{pmatrix} 1 - \frac{L}{f} - \frac{L^2}{f^2} & 2L + \frac{L^2}{f} \\ -\frac{L}{f^2} & 1 + \frac{L}{f} \end{pmatrix}
\]
THIN LENS APPROXIMATION AND FODO CELL (3/4)

- By comparison with the general form of a Twiss matrix

\[
M(s) = \begin{bmatrix}
\cos \mu + \alpha(s) \sin \mu & \beta(s) \sin \mu \\
- \gamma(s) \sin \mu & \cos \mu - \alpha(s) \sin \mu
\end{bmatrix}
\]

one obtains

\[
\beta_{QF} \sin \mu = 2L + \frac{L^2}{f} = 2L \left(1 + \frac{L}{2f}\right)
\]

and

\[
\text{Tr} \left[ M(s) \right] = 2 \cos \mu = 2 - \frac{L^2}{f^2} \Rightarrow \frac{L}{2f} = \sin \left(\frac{\mu}{2}\right)
\]

\[
\Rightarrow \beta_{QF} = 2L \frac{1 + \sin \left(\frac{\mu}{2}\right)}{\sin \mu}
\]
The betatron function at the defocusing quadrupole QD can be found by considering the DOFO cell instead of a FODO cell

\[
\beta_{QD} = 2L \frac{1 - \sin\left(\frac{\mu}{2}\right)}{\sin\mu}
\]

- Stability of the FODO cell requires \(|\cos\mu| \leq 1\)

\[|f| \geq \frac{L}{2}\]

\(<=>\) the stability of the FODO cell is thus obtained for distances \(L\) between the quadrupoles up to twice their focal length
An invariant i.e. a constant of motion, (called COURANT-SNYDER INVARIANT) can be found from the solution of the Hill’s equation

\[ u(s) = a \sqrt{\beta(s)} \cos[\mu(s) - \varphi] \]

\( \Rightarrow \) Equation of an ellipse (motion for one particle) in the phase space plane \((u, u')\), with area \(\pi a^2\)

\[
\gamma(s) u(s)^2 + 2 \alpha(s) u(s) u'(s) + \beta(s) u'(s)^2 = a^2
\]

“Single-particle” emittance: \(\varepsilon = a^2\)
The shape and orientation of the phase plane ellipse evolve along the machine (depending on the Twiss parameters), but not its area.
Transformation rule for phase ellipses through the lattice

\[
\begin{bmatrix}
  u(s) \\
  u'(s)
\end{bmatrix} =
\begin{bmatrix}
  C(s) & S(s) \\
  C'(s) & S'(s)
\end{bmatrix}
\begin{bmatrix}
  u(s_0) \\
  u'(s_0)
\end{bmatrix}
\]

where \( C(s) \) and \( S(s) \) are 2 independent solutions of Hill’s equation satisfying the particular initial conditions: \( C(s_0) = S'(s_0) = 1 \) and \( C'(s_0) = S(s_0) = 0 \)

For \( K(s) = K > 0 \):

\[
C(s) = \cos\left(\sqrt{K} \ s\right)
\]

\[
S(s) = \frac{\sin\left(\sqrt{K} \ s\right)}{\sqrt{K}}
\]

\[
\begin{bmatrix}
  \beta(s) \\
  \alpha(s) \\
  \gamma(s)
\end{bmatrix} =
\begin{bmatrix}
  C(s)^2 & -2S(s)C(s) & S(s)^2 \\
  -C(s)C'(s) & S'(s)C(s) + S(s)C'(s) & -S(s)S'(s) \\
  C'(s)^2 & -2S'(s)C'(s) & S'(s)^2
\end{bmatrix}
\begin{bmatrix}
  \beta(s_0) \\
  \alpha(s_0) \\
  \gamma(s_0)
\end{bmatrix}
\]
Stroboscopic representation or POINCARÉ MAPPING

Depends on the TUNE (reminder: number of betatron oscillations per machine revolution)
BEAM EMITTANCE, ENVELOPE AND DIVERGENCE (5/7)

- BEAM EMITTANCE = Measure of the spread in phase space of the points representing beam particles ⇒ 3 definitions

1) In terms of the phase plane “amplitude” \( a_q \) enclosing \( q \% \) of the particles

\[
\int \int dx \, dx' = \pi \, \varepsilon_x^{(q\%)}
\]

ellipse of "amplitude" \( a_q \)  

\[\text{[mm mrad]} \text{ or } [\mu\text{m}]\]

2) In terms of the 2\textsuperscript{nd} moments of the particle distribution

\[
\varepsilon_x^{(\text{stat})} = \sqrt{<x^2> <x'^2> - <xx'>^2}
\]

Determinant of the covariance matrix

3) In terms of \( \sigma_x \) the standard deviation of the particle distribution in real space (= projection onto the \( x \)-axis)

\[
\varepsilon_x^{(\sigma_x)} = \frac{\sigma_x^2}{\beta_x}
\]
The \( \beta \)-function reflects the size of the beam and depends only on the lattice.
BEAM EMITTANCE, ENVELOPE AND DIVERGENCE (7/7)

- MACHINE mechanical (i.e. from the vacuum chamber) ACCEPTANCE or APERTURE = Maximum beam emittance

- NORMALIZED BEAM EMITTANCE

\[
\varepsilon_{x,\text{norm}}(\sigma_x) = \beta_r \gamma_r \varepsilon_x(\sigma_x)
\]

⇒ The normalized emittance is conserved during acceleration (in the absence of collective effects…)

- ADIABATIC DAMPING: As \( \beta_r \gamma_r \) increases proportionally to the particle momentum \( p \), the (physical) emittance decreases as \( 1 / p \)

- However, many phenomena may affect (increase) the emittance

- An important challenge in accelerator technology is to preserve beam emittance and even to reduce it (by COOLING)
In practice, particle beams have a finite dispersion of momenta about the ideal momentum $p_0$. A particle with momentum $p \neq p_0$ will perform betatron oscillations around a different closed orbit from that of the reference particle.

\[
x'' + \left( K_0 + \frac{1}{\rho_x^2} \right) x = \frac{\delta}{\rho_x} \\
y'' - \left( K_0 - \frac{1}{\rho_y^2} \right) y = \frac{\delta}{\rho_y}
\]

\[
\Rightarrow u'' + K(s) \ u = \frac{\delta}{\rho(s)}
\]

\[
\Rightarrow u(s) = u_\delta(s) + u_\beta(s)
\]

Displacement of the closed orbit for the off-momentum particle from the on-momentum particle.

Betatron oscillation around the off-momentum orbit.
**DISPERSION (2/10)**

\[
u_\delta(s) = D(s) \delta
\]

\(D(s)\) is called the **DISPERSION FUNCTION**

- The dispersion function satisfies the equation

\[
D'' + K(s) D = \frac{1}{\rho(s)}
\]

and can be found to be given by

\[
D(s) = \frac{\sqrt{\beta(s)}}{2 \sin(\pi Q)} \int_s^{s+C} dt \frac{\sqrt{\beta(t)}}{\rho(t)} \cos \left[ \mu(t) - \mu(s) - \pi Q \right]
\]
DISPERSION (3/10)

=> i) The dispersion function depends on all the bending magnets in the accelerator

ii) To get stable off-momentum orbits, the machine tune $Q$ must not be an integer (otherwise resonance effect will occur)

- Applying the same method as the one used for betatron oscillations, the dispersion function can be derived by applying the matrix formalism (instead of using the previous formula) =>

$$
\begin{bmatrix}
u(s) \\
u'(s) \\
\delta
\end{bmatrix} = M \begin{bmatrix} s / s_0 \\
\delta
\end{bmatrix} \begin{bmatrix}
u(s_0) \\
u'(s_0) \\
\delta
\end{bmatrix}
$$

$$
M(s_0) \equiv M(s_0 + L / s_0) = M_N M_{N-1} \cdots M_2 M_1
$$

$$
= \begin{bmatrix} m_{11}(s_0) & m_{12}(s_0) & m_{13}(s_0) \\
m_{21}(s_0) & m_{22}(s_0) & m_{23}(s_0) \\
0 & 0 & 1
\end{bmatrix}
$$

Usual 2 × 2 transfer matrix for betatron oscillations
It can be shown that

\[
\begin{bmatrix}
D(s) \\
D'(s) \\
1
\end{bmatrix} = 
\begin{bmatrix}
m_{11}(s_0) & m_{12}(s_0) & m_{13}(s_0) \\
m_{21}(s_0) & m_{22}(s_0) & m_{23}(s_0) \\
0 & 0 & 1
\end{bmatrix} 
\begin{bmatrix}
D(s) \\
D'(s) \\
1
\end{bmatrix}
\]

\[
D(s_0) = \frac{m_{13}(s_0)[1 - m_{22}(s_0)] + m_{12}(s_0)m_{23}(s_0)}{2 - m_{11}(s_0) - m_{22}(s_0)}
\]

\[
D'(s_0) = \frac{m_{23}(s_0)[1 - m_{11}(s_0)] + m_{21}(s_0)m_{13}(s_0)}{2 - m_{11}(s_0) - m_{22}(s_0)}
\]

=> The dispersion function and its derivative can be found at any location \(s_0\) around the ring
Example of a dipole sector magnet

\[ K(s) = \frac{1}{\rho_0^2} \]

\[ \vartheta = \frac{l}{\rho_0} \]

\[ \Rightarrow \]

\[ C(s) = \cos \left( \frac{s - s_0}{\rho_0} \right) \]

\[ S(s) = \rho_0 \sin \left( \frac{s - s_0}{\rho_0} \right) \]
Note that the $3 \times 3$ transfer matrix for synchrotron magnets (= combined-function dipole-quadrupole magnets) can be derived similarly, replacing $\frac{1}{\rho_0}$ by $\sqrt{K}$ where

$$K = K_0 + \frac{1}{\rho_0^2}$$
Example of the thin-lens FODO lattice with cell length $2L$, in which the drift spaces are replaced by dipole sector magnets of length $L$ and bending radius $\rho_0$.

\[
M_{F_DipDip} \left( \frac{2L}{0} \right) = M_{dip} M_{QD} M_{dip} M_{QF}
\]

\[
M_{QF,D} = \begin{bmatrix}
1 & 0 & 0 \\
\mp f^{-1} & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
M_{dip} = \begin{bmatrix}
1 & L & \frac{L^2}{2\rho_0} \\
0 & 1 & \frac{L}{\rho_0} \\
0 & 0 & 1
\end{bmatrix}
\]
Using the definition of the dispersion function, it can be computed at the position \( s_0 \) (of the focusing quad QF)

\[
D_{QF} = \frac{4f^2}{\rho_0} \left(1 + \frac{L}{4f} \right)
\]

Similarly, the dispersion in the defocusing quad QD is derived by inverting the QF and QD in the cell

\[
D_{QD} = \frac{4f^2}{\rho_0} \left(1 - \frac{L}{4f} \right)
\]
Example of LHC optics for the Interaction Point 5 (CMS) in collision

MAD program

LHC V6.500 Collision LHCb1 IR5 Crossing Bumps(IP1=100PERCENT IP5=100PERCENT IP2=100PERCENT IP3=100PERCENT)

\[ \beta_x, \beta_y, D_x, D_y \]

Momentum offset = 0

\[ \text{s [km]} \]
BETATRON MATCHING = The phase space ellipses at the injection (ejection) point of the circular machine, and the exit (entrance) of the beam transport line, should be homothetic. To do this, the Twiss parameters are modified using quadrupoles. If the ellipses are not homothetic, there will be a dilution (i.e. a BLOW-UP) of the emittance.

DISPERSION MATCHING = $D_x$ and $D'_x$ should be the same at the injection (ejection) point of the circular machine, and the exit (entrance) of the beam transport line. If there are different, there will be also a BLOW-UP, but due to a missteering (because the beam is not injected on the right orbit).
NORMALIZED FLOQUET’S COORDINATES (1/3)

- Normalized (Floquet’s) coordinates

\[
\eta = \frac{x}{\sqrt{\beta_x(s)}} \quad \xi = \frac{y}{\sqrt{\beta_y(s)}} \quad \phi = \frac{1}{Q_{x,y}} \int_{s_0}^{s} \frac{dt}{\beta_{x,y}(t)}
\]

=> Perturbed (e.g. horizontal) Hill’s equation

\[
x'' + K(s) x = p(x,y,s)
\]

becomes

\[
\ddot{\eta} + Q^2 \eta = Q^2 \beta^{3/2} p(\eta,\phi)
\]

\[
\dot{\eta} = \frac{d\eta}{d\phi}
\]
NORMALIZED FLOQUET’S COORDINATES (2/3)

- When the perturbation vanishes =>
  \[ \ddot{\eta} + Q^2 \eta = 0 \]

  \[ \eta(\phi) = a \cos \left[ \mu(\phi) - \phi \right] \]

  \[ \mu(\phi) = Q \phi = \int_{s_0}^{s} \frac{dt}{\beta(t)} \]

- Phase advance

- Particle trajectory in phase space
  \[ \eta^2 + \left( \frac{d\eta}{d\mu} \right)^2 = a^2 \]
The particle trajectory in the phase plane \((\eta, \frac{d\eta}{d\mu})\) is thus a circle of radius equal to the amplitude \(a\) of the oscillation.

The phase advances by \(2\pi\) every betatron oscillation or by \(2\pi Q\) every machine revolution.
CHROMATICITY AND ITS CORRECTION (1/17)

- Chromaticity
  - Equations of motion of an off-momentum particle (without skew magnets and sextupoles and in a zero-curvature region)
    
    \[
    x'' + K_0 \left( 1 - \delta \right) x = 0
    \]
    
    \[
    y'' - K_0 \left( 1 - \delta \right) y = 0
    \]

  - Chromaticity = Variation of the tune with the momentum
    
    \[
    Q' = \frac{\Delta Q}{\delta}
    \]
    
  - Relative chromaticity
    
    \[
    \xi = \frac{Q'}{Q}
    \]
Once the particles are spread by momentum in a region with dispersion, we can apply focusing corrections depending on the momentum using a sextupole magnet.

**SEXTUPOLE = 1st nonlinear magnet**
The sextupole is focusing for the higher-momentum particles and defocusing for the lower-momentum particles. It can be used to correct the chromatic focusing errors in a region with non-zero dispersion.

Let’s consider the equations of motion including the contributions from the regular sextupole magnets:

\[ x'' + K_0 \left( 1 - \delta \right) x = -\frac{1}{2} S_0 \left( x^2 - y^2 \right) \]

\[ y'' - K_0 \left( 1 - \delta \right) y = S_0 \ x \ y \]

Using \( x(s) = x_\delta(s) + x_\beta(s) \) and \( y(s) = y_\beta(s) \) yields


CHROMATICITY AND ITS CORRECTION (4/17)

\[ x''_\beta + K_0 x_\beta = \left( K_0 - S_0 D_x \right) x_\beta \delta \]

\[ y''_\beta - K_0 y_\beta = -\left( K_0 - S_0 D_x \right) y_\beta \delta \]

(discarding the terms which do not depend on the betatron motion, because they do not contribute to the chromatic tune shift, and ignoring the non-chromatic terms of 2\textsuperscript{nd} order)

• Introducing the normalized coordinates, yields
**CHROMATICITY AND ITS CORRECTION (5/17)**

\[ \ddot{\eta} + Q_x^2 \eta = Q_x p_{x\delta} (\phi) \eta \]

\[ \ddot{\xi} + Q_y^2 \xi = -Q_y p_{y\delta} (\phi) \xi \]

\[ p_{x\delta} (\phi) = Q_x \beta_x^2 \left[ K_0 (\phi) - S_0 (\phi) D_x \right] \delta \]

\[ p_{y\delta} (\phi) = Q_y \beta_y^2 \left[ K_0 (\phi) - S_0 (\phi) D_x \right] \delta \]

Periodic functions with period 2 \( \pi \)
Expanding the perturbations in Fourier series yields

\[ p_{x\delta}(\phi) = \sum_{m=-\infty}^{+\infty} \hat{p}_{x\delta}(m) e^{jm\phi} \]

\[ = \hat{p}_{x\delta}(0) + \sum_{m\neq 0} \hat{p}_{x\delta}(m) e^{jm\phi} \]

with

\[ \hat{p}_{x\delta}(m) = \frac{1}{2\pi} \int_{0}^{2\pi} d\phi \ p_{x\delta}(\phi) e^{-jm\phi} \]
CHROMATICITY AND ITS CORRECTION (7/17)

\[ \dot{\eta} + \tilde{Q}_x^2 (\phi) \eta = 0 \]

with

\[ \tilde{Q}_x^2 (\phi) = Q_x^2 \left[ 1 - \frac{\hat{P}_{x\delta}(0)}{Q_x} \right] - Q_x \sum_{m \neq 0} \hat{P}_{x\delta}(m) e^{jm\phi} \]

Averages to 0 over 1 period \(2\pi\)

- Averaging over a period gives the static tune shift due to chromatic effect

\[ \ddot{\eta} + (Q_x + \Delta Q_x)^2 \eta = 0 \]

with

\[ \Delta Q_x = -\frac{\hat{P}_{x\delta}(0)}{2} \]
\[
\Delta Q_x = -\frac{1}{4\pi} \int_0^{2\pi} d\phi \ p_x \delta (\phi)
\]

\[
= -\frac{\delta}{4\pi} \int_{s_0}^{s_0+C} ds \beta_x (s) \left[ K_0 (s) - S_0 (s) D_x (s) \right]
\]

\[
\Delta Q_y = \frac{\delta}{4\pi} \int_{s_0}^{s_0+C} ds \beta_y (s) \left[ K_0 (s) - S_0 (s) D_x (s) \right]
\]

\[
\xi_x = -\frac{1}{4\pi Q_x} \int_s^{s+C} dt \beta_x (t) \left[ K_0 (t) - S_0 (t) D_x (t) \right]
\]

\[
\xi_y = +\frac{1}{4\pi Q_y} \int_s^{s+C} dt \beta_y (t) \left[ K_0 (t) - S_0 (t) D_x (t) \right]
\]
The contribution to chromaticity arising from pure quadrupole elements (and also pure dipoles) is called natural chromaticity. The natural chromaticities of a lattice are:

\[
\xi_x^{\text{natural}} = - \frac{1}{4 \pi Q_x} \int_s^{s+C} dt \beta_x(t) K_0(t)
\]

\[
\xi_y^{\text{natural}} = + \frac{1}{4 \pi Q_y} \int_s^{s+C} dt \beta_y(t) K_0(t)
\]
Natural chromaticity of a FODO cell

\[
\xi_{\text{natural}} = -\frac{1}{4\pi Q_x} \left( \frac{1}{(\beta_{Q_F} K_{Q_F}^2 + \beta_{Q_D} K_{Q_D}^2)(1 + \beta_{Q_D} K_{Q_D})} \right)
\]

\[
\frac{1}{4\pi Q_x} \int_{s+c}^{s} \left[ \beta_{Q_F} \int ds K_{Q_F} \right] dt \beta_x (t) \left( K_0(t) \right) d s_{K_{Q_D}} + \beta_{Q_D} \int ds K_{Q_D}
\]

\[
\xi_{\text{natural}} = -\frac{1}{4\pi Q_x} \left( \frac{1}{(\beta_{Q_F} K_{Q_F}^2 + \beta_{Q_D} K_{Q_D}^2)(1 + \beta_{Q_D} K_{Q_D})} \right)
\]

CHROMATICITY AND ITS CORRECTION (10/17)
CHROMATICITY AND ITS CORRECTION (11/17)

\[ \xi_{\text{natural}}^x = - \frac{1}{4 \pi Q_x f} \left( \beta_x^{QF} - \beta_x^{QD} \right) \]

\[ \frac{1}{f} = K^{QF} \quad l = - K^{QD} \ l \]

- We saw previously that

\[ \beta_x^{QF} = 2 \ L \frac{1 + \sin \left( \frac{\mu}{2} \right)}{\sin \mu} \]

\[ \beta_x^{QD} = 2 \ L \frac{1 - \sin \left( \frac{\mu}{2} \right)}{\sin \mu} \]

\[ \frac{1}{f} = \frac{2}{L} \ \sin \left( \frac{\mu}{2} \right) \]

\[ \Rightarrow \]

\[ \xi_{\text{natural}}^x = - \frac{1}{\pi Q_x} \tan \left( \frac{\mu}{2} \right) \]
If the full lattice is composed of $N_{\text{cell}}$ similar FODO cells

Furthermore,

$$Q_x = \frac{N_{\text{cell}} \mu}{2 \pi}$$

$$\xi_{\text{natural}}^x = - \frac{N_{\text{cell}}}{\pi Q_x} \tan\left(\frac{\mu}{2}\right)$$

Similarly

$$\xi_{\text{natural}}^y = - \frac{2}{\mu} \tan\left(\frac{\mu}{2}\right)$$
The natural chromaticities of a synchrotron made up of FODO cells are negative. For instance, if the phase advance per cell is $\pi / 2$, then

$$\xi_x^{\text{natural}} = \xi_y^{\text{natural}} = -\frac{4}{\pi} \approx -1.3$$
More generally, the natural chromaticities are always negative since for the higher-momentum particles the focusing is less effective and then the tune is reduced.

The control of the chromaticity (using SEXTUPOLE magnets) is very important for 2 reasons:

- Avoid shifting the beam on resonances due to changes induced by chromatic effects
- Prevent some transverse coherent (head-tail) instabilities

=> CHROMATICITY CORRECTION / CONTROL USUALLY REQUIRED
**Chromaticity AND ITS CORRECTION (15/17)**

- **Chromaticity correction**
  - The chromaticity equations suggest the insertion of sextupoles close to each quadrupole, where the dispersion function is non-zero, in order to correct the chromaticity.

\[
\xi_x = - \frac{1}{4\pi Q_x} \int_{s}^{s+C} dt \beta_x(t) \left[ K_0(t) - S_0(t) D_x(t) \right]
\]

- However, such localized corrections are seldom feasible.
CHROMATICITY AND ITS CORRECTION (16/17)

- A standard way of adjusting both the horizontal and vertical chromaticities is to use families of sextupoles with moderate strength, distributed around the ring

\[
\xi_x = \xi_x^{\text{natural}} + \frac{1}{4\pi Q_x} \int_s^{s+C} dt \beta_x(t) S_0(t) D_x(t)
\]

\[
\xi_y = \xi_y^{\text{natural}} - \frac{1}{4\pi Q_y} \int_s^{s+C} dt \beta_y(t) S_0(t) D_x(t)
\]

- Using the thin-lens approximation, one can write

\[
\xi_x = \xi_x^{\text{natural}} + \frac{1}{4\pi Q_x} \sum_{i=1}^{N} dt \beta_{xi} S_{0i} D_{xi} l_{si}
\]

\[
\xi_y = \xi_y^{\text{natural}} - \frac{1}{4\pi Q_y} \sum_{i=1}^{N} dt \beta_{yi} S_{0i} D_{xi} l_{si}
\]
The sextupole strengths are obtained by solving a linear system of equations. Assuming only 2 sextupoles in the ring, yields

\[
S_{01} = - \frac{4 \pi}{D_{x1} l_s} \left( \frac{b_{y2} Q_x \Delta \xi_x + b_{x2} Q_y \Delta \xi_y}{b_{x1} b_{y2} - b_{x2} b_{y1}} \right)
\]

\[
S_{02} = \frac{4 \pi}{D_{x2} l_s} \left( \frac{b_{y1} Q_x \Delta \xi_x + b_{x1} Q_y \Delta \xi_y}{b_{x1} b_{y2} - b_{x2} b_{y1}} \right)
\]

Assuming the same length for both sextupoles, \( l_s \).
In the presence of extra (NONLINEAR) FORCES, the Hill’s equation takes the general form

\[ x''(s) + K_x(s) x(s) = P_x(x, y, s) \]

Any perturbation

Perturbation terms in the equation of motion may lead to UNSTABLE motion, called RESONANCES, when the perturbing field acts in synchronism with the particle oscillations.

A multipole of \( n \)th order is said to generate resonances of order \( n \). Resonances below the 3\textsuperscript{rd} order (i.e. due to dipole and quadrupole field errors for instance) are called LINEAR RESONANCES. The NONLINEAR RESONANCES are those of 3\textsuperscript{rd} order and above.
Nonlinearities and Resonances (2/5)

- **General Resonance Conditions**

  \[ M Q_x + N Q_y = P \]

  where \( M, N \) and \( P \) are integers, \( P \) being non-negative, \(|M| + |N|\) is the order of the resonance and \( P \) is the order of the perturbation harmonic.

- Plotting the resonance lines for different values of \( M, N, \) and \( P \) in the \((Q_x, Q_y)\) plane yields the so-called Resonance or Tune Diagram.

This dot in the tune diagram is called the **Working Point** (case of the PS, here).
NONLINEARITIES AND RESONANCES (3/5)

- RESONANCE WIDTH = Band with some thickness around every resonance line in the resonance diagram, in which the motion may be unstable, depending on the oscillation amplitude

- STOPBAND = Resonance width when the resonance is linear (i.e. below the 3rd order), because the entire beam becomes unstable if the operating point \((Q_x, Q_y)\) reaches this region of tune values

- DYNAMIC APERTURE = Largest oscillation amplitude which is stable in the presence of nonlinearities

- TRACKING: In general, the equations of motion in the presence of nonlinear fields are untractable for any but the simplest situations. Tracking consists to simulate (using computer programs such as MAD) particle motion in circular accelerators in the presence of nonlinear fields
NONLINEARITIES AND RESONANCES (4/5)

- **KICK MODEL:** Any nonlinear magnet is treated in the “point-like” approximation (i.e. the particle position is assumed not to vary as the particle traverses the field), the motion in all other elements of the lattice is assumed to be linear. Thus, at each turn the local magnetic field gives a “kick” to the particle, deflecting it from its unperturbed trajectory.

- **HENON MAPPING**
  = Stroboscopic representation of phase-space trajectories (normalised ⇒ circles instead of ellipses for linear motion) on every machine turn at the fixed azimuthal position of the perturbation.

```latex
\begin{align*}
Q_x &= 0.324 & x'_\text{norm} \\
Q_x &= 0.320 \\
Q_x &= 0.252 \\
Q_x &= 0.211
\end{align*}
```

Close to 1/3

Close to 1/4

Close to 1/5
SEPARATRICES define boundaries between stable motion (bounded oscillations) and unstable motion (expanding oscillations).

The 3rd order resonance is a drastic (unstable) one because the particles which go onto this resonance are lost.

(STABLE) ISLANDS: For the higher order resonances (e.g. 4th and 5th) stable motions are also possible in (stable) islands. There are 4 stable islands when the tune is closed to a 4th order resonance and 5 when it is closed to a 5th order resonance.