LUMINOSITY

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- Colliders vs. fixed-target experiments
- General definition of luminosity
- Simplest formula for Head-On (HO) collisions
- Some complications
  - Crossing angle
  - Transverse beam offset
  - Hourglass effect
- Integrated luminosity and maximization
- Pile-up, luminosity leveling and luminous region
- Summary: How to reach high luminosity?
COLLIDERS VS. FIXED-TARGET EXPERIMENTS (1/2)

- Using the relativistic equations given in Introduction, it can be seen that

\[
\sqrt{s} = E_{CM} = \sqrt{m_{01}^2 c^4 + m_{02}^2 c^4 + 2 \left( E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2 c^2 \right)}
\]

- For a fixed target (\(\vec{p}_2 = 0\)) and if we neglect the masses (i.e. if we are at sufficiently high energy)

\[
\sqrt{s} = \sqrt{2 E_1 m_{02} c^2}
\]

- For a collider (\(\vec{p}_2 = -\vec{p}_1\)) and if we neglect the masses (i.e. if we are at sufficiently high energy)

\[
\sqrt{s} = 2 E_1
\]
Numerical applications (for the available energy in the CM, i.e. to create new particles)

- LHC ($p^+p^+$, 7 TeV / beam)
  - Collider mode => 14 TeV
  - Fixed-target mode => ~ 115 GeV (i.e. ~ 122 times less)

- LEP ($e^+e^-$, 105 GeV / beam)
  - Collider mode => 210 GeV
  - Fixed-target mode => ~ 0.3 GeV (i.e. 626 times less)
By definition, the luminosity $L$ is the time-averaged integral over the interaction volume $\Omega$ of the number of reactions per unit time and volume.

$$L = \frac{1}{\sigma_r T_b} \int_0^{T_b} \int_{\Omega} \frac{d^2 N}{d t d \Omega} d t d \Omega$$

- Total cross section of the reaction $[m^2]$
- Inverse of the bunch collision frequency $[s]$
- Interaction volume where the 2 beams collide $[m^3]$: $d\Omega = ds dx dy$
The number of reactions per unit time and unit volumes satisfies the following relation associated with the Lorentz transformation of the variables => Luminosity density \( S \)

\[
S = \frac{1}{\sigma_r} \frac{d^2 N}{dt d\Omega} = N_1 N_2 \rho_1(x, y, s, t) \rho_2(x, y, s, t) M_{KLF}
\]

- Density of bunch 1
- Density of bunch 2
- Møller Kinematic Luminosity Factor
- Numbers of particles / bunch (1 and 2)
- Correction factor that makes \( S \) a relativistic invariant

\[
M_{KLF} = \sqrt{\left( \vec{v}_1 - \vec{v}_2 \right)^2 - \left( \frac{\vec{v}_1 \times \vec{v}_2}{C^2} \right)^2}
\]

\[
\int d\Omega \rho_1 = 1 \quad \int d\Omega \rho_2 = 1
\]
GENERAL DEFINITION OF LUMINOSITY (3/7)

- Møller Kinematic Luminosity Factor is linked to the relative velocity between the 2 beams $v_{21}$ (see Introduction)

$$M_{KLF} = v_{21} \left( 1 - \frac{\vec{v}_1 \cdot \vec{v}_2}{c^2} \right)$$

- $\vec{v}_1$ = Velocity in the Laboratory frame of all particles of bunch 1
- $\vec{v}_2$ = Velocity in the Laboratory frame of all particles of bunch 2

$$L = M N_1 N_2 f_{rev} M_{KLF} \int_0^{T_b} \int_\Omega \rho_1(x, y, s, t) \rho_2(x, y, s, t) \, dt \, d\Omega$$
GENERAL DEFINITION OF LUMINOSITY (4/7)

- Collision without crossing angle
  \[ S_0 = c \cdot t \]

- Collision with crossing angle (general case)

\[ L = MN_1 N_2 f_{rev} \frac{M_{KLF}}{c} \int \int \int \int \rho_1(x, y, s, -s_0) \rho_2(x, y, s, s_0) \, dx \, dy \, ds \, ds_0 \]
Møller Kinematic Luminosity Factor

(general case with crossing angle)

\[ M_{KLF} = \sqrt{\left( \vec{v}_1 - \vec{v}_2 \right)^2 - \left( \frac{\vec{v}_1 \times \vec{v}_2}{c^2} \right)^2} \]

\[ \vec{v}_1 = \begin{pmatrix} v_1 \sin \frac{\Phi}{2} \\ 0 \\ v_1 \cos \frac{\Phi}{2} \end{pmatrix} \]

\[ \vec{v}_2 = \begin{pmatrix} v_2 \sin \frac{\Phi}{2} \\ 0 \\ -v_2 \cos \frac{\Phi}{2} \end{pmatrix} \]

\[ v_1 = \beta_1 c \]

\[ v_2 = \beta_2 c \]
### General Definition of Luminosity (6/7)

\[ \vec{v}_1 - \vec{v}_2 = \begin{vmatrix} \frac{(v_1 - v_2) \sin \Phi}{2} \\ 0 \\ \frac{(v_1 + v_2) \cos \Phi}{2} \end{vmatrix} \]

and

\[ \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} 0 \\ \frac{v_1 v_2 \cos \Phi}{2} \\ \frac{\sin \Phi}{2} \end{vmatrix} \]

\[ M_{KLF} = \sqrt{v_1^2 + v_2^2 + 2v_1 v_2 \cos \Phi - \frac{v_1^2 v_2^2}{c^2} \sin^2 \Phi} \]

\[ v_1 = \beta_1 c \]
\[ v_2 = \beta_2 c \]

\[ \frac{M_{KLF}}{c} = \sqrt{\beta_1^2 + \beta_2^2 + 2 \beta_1 \beta_2 \cos \Phi - \beta_1^2 \beta_2^2 \sin^2 \Phi} \]
GENERAL DEFINITION OF LUMINOSITY (7/7)

- If $\beta_1 = \beta_2 = \beta$ =>
  $$\frac{M_{KLF}}{c} = 2 \beta \cos \frac{\Phi}{2} \sqrt{1 - \beta^2 \sin^2 \frac{\Phi}{2}}$$

- If $\beta_1 = \beta_2 = \beta = 1$ =>
  $$\frac{M_{KLF}}{c} = 2 \cos^2 \frac{\Phi}{2}$$

- If $\Phi = 0$ =>
  $$\frac{M_{KLF}}{c} = 2$$
Luminosity in the absence of crossing angle (and transverse beam offset and hourglass effect)

\[
L = M N_1 N_2 f_{\text{rev}} 2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho_1(x, y, s, -s_0) \rho_2(x, y, s, s_0) \, dx \, dy \, ds \, ds_0
\]

- Assuming that the densities are uncorrelated in all planes

\[
\rho_1(x, y, s, -s_0) = \rho_{1x}(x) \rho_{1y}(y) \rho_{1s}(s - s_0)
\]

\[
\rho_2(x, y, s, s_0) = \rho_{2x}(x) \rho_{2y}(y) \rho_{2s}(s + s_0)
\]

- Assuming Gaussian distributions in all dimensions

\[
\rho_{1x}(x) = \frac{1}{\sigma_{1x} \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_{1x}^2}}, \ldots
\]
SIMPLEST FORMULA FOR HEAD-ON COLLISIONS (2/4)

\[ L = \frac{2 M N_1 N_2 f_{rev}}{(2 \pi)^3 \sigma_{1x} \sigma_{2x} \sigma_{1y} \sigma_{2y}} \]

\[ \int \int \int \int e^{-\frac{x^2}{2\sigma_{1x}^2}} e^{-\frac{y^2}{2\sigma_{1y}^2}} e^{-\frac{(s-s_0)^2}{2\sigma_{1s}^2}} e^{-\frac{x^2}{2\sigma_{2x}^2}} e^{-\frac{y^2}{2\sigma_{2y}^2}} e^{-\frac{(s+s_0)^2}{2\sigma_{2s}^2}} d x d y d s d s_0 \]

Assuming \( \sigma_{1s} = \sigma_{2s} = \sigma_s \)

\[ e^{-\frac{s^2}{2\sigma_s^2}} e^{-\frac{s_0^2}{2\sigma_s^2}} \]

\( \int \int e^{-\frac{s^2}{2\sigma_s^2}} e^{-\frac{s_0^2}{2\sigma_s^2}} d s d s_0 = \pi \)

\( \Rightarrow \)

\( L = \frac{M N_1 N_2 f_{rev}}{4 \pi^2 \sigma_{1x} \sigma_{2x} \sigma_{1y} \sigma_{2y}} \int \int e^{-\frac{x^2}{2\sigma_{1x}^2}} e^{-\frac{y^2}{2\sigma_{1y}^2}} e^{-\frac{x^2}{2\sigma_{2x}^2}} e^{-\frac{y^2}{2\sigma_{2y}^2}} d x d y \)

(see Useful relations in Introduction)
SIMPLEST FORMULA FOR HEAD-ON COLLISIONS (3/4)

- Assuming \( \sigma_{1x} = \sigma_{2x} = \sigma_x \)
  \( \sigma_{1y} = \sigma_{2y} = \sigma_y \)

  one finally obtains

\[
L = \frac{M N_1 N_2 f_{rev}}{4 \pi \sigma_x \sigma_y}
\]

Let’s call it \( L_0 \)

- If \( \sigma_{1x} \neq \sigma_{2x} \)
  \( \sigma_{1y} \neq \sigma_{2y} \)

\[
L = \frac{M N_1 N_2 f_{rev}}{2 \pi \sqrt{\sigma_{1x}^2 + \sigma_{2x}^2} \sqrt{\sigma_{1y}^2 + \sigma_{2y}^2}}
\]
Assuming

\[ \sigma_{1x} = \sigma_{2x} = \sigma_x \]
\[ \sigma_{1y} = \sigma_{2y} = \sigma_y \]
\[ \sigma_x = \sigma_y = \sigma \]
\[ N_1 = N_2 = N_b \]

\[ L_0 = \frac{M N_b^2 f_{\text{rev}} \beta \gamma}{4 \pi \beta^* \epsilon_n} \]

\[ \epsilon_n = \beta \gamma \epsilon = \beta \gamma \frac{\sigma^2}{\beta^*} \]

Normalized transverse beam emittance

\[ \beta\text{-function at the collision point} \]

Numerical application for LHC => \( L_0 = 1.2 \times 10^{34} \text{ cm}^{-2}\text{s}^{-1} \)
SOME COMPLICATIONS: CROSSED ANGLE (1/5)

- Luminosity in the presence of crossing angle (in the x-s plane)

- Rotation of $\frac{\Phi}{2}$ for beam 1

- Rotation of $-\frac{\Phi}{2}$ for beam 2
SOME COMPLICATIONS: CROSSING ANGLE (2/5)

The following relations are thus obtained (see also Useful relations in Introduction)

\[
\begin{align*}
(x_1) &= \left[ \cos \frac{\Phi}{2} \quad -\sin \frac{\Phi}{2} \right] \begin{pmatrix} x \\ s_1 \end{pmatrix} \\
(s_1) &= \left[ \sin \frac{\Phi}{2} \quad \cos \frac{\Phi}{2} \right] \begin{pmatrix} x \\ s \end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
(x_2) &= \left[ \cos \frac{\Phi}{2} \quad \sin \frac{\Phi}{2} \right] \begin{pmatrix} x \\ s_2 \end{pmatrix} \\
(s_2) &= \left[ -\sin \frac{\Phi}{2} \quad \cos \frac{\Phi}{2} \right] \begin{pmatrix} x \\ s \end{pmatrix}
\end{align*}
\]

\[
L_{CA} = 2 \cos^2 \frac{\Phi}{2} M N_1 N_2 f_{rev}
\]

\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho_{1x}(x_1) \rho_{1y}(y_1) \rho_{1s}(s_1 - s_0) \rho_{2x}(x_2) \rho_{2y}(y_2) \rho_{2s}(s_2 + s_0) \, dx \, dy \, ds \, ds_0
\]
SOME COMPLICATIONS: CROSSING ANGLE (3/5)

- Assuming

\[ \sigma_{1x} = \sigma_{2x} = \sigma_x \]
\[ \sigma_{1y} = \sigma_{2y} = \sigma_y \]
\[ \sigma_{1s} = \sigma_{2s} = \sigma_s \]
\[ y_1 = y_2 = y \]

Useful relations in Introduction

\[ \int_{-\infty}^{\infty} e^{-\left(at^2 + bt + c\right)} dt = \sqrt{\frac{\pi}{a}} \frac{b^2}{4a - c} \]

yields

\[ L_{CA} = L_0 F_{CA} \]

- Numerical application for LHC

\[ \Rightarrow F_{CA} = 0.84 \]

\[ F_{CA} = \frac{1}{\sqrt{1 + \left(\frac{\sigma_s}{\sigma_x} \tan\frac{\Phi}{2}\right)^2}} \]

\[ \tan\frac{\Phi}{2} \sim \frac{\Phi}{2} \]
SOME COMPLICATIONS: CROSSING ANGLE (4/5)

$Lumi_{reduction} from crossing angle$

$LHC nominal: 7.5 \text{ cm}$
Possibility to compensate for this geometric loss factor => Use “Crab Cavities”

=> Already used in leptons machine (KEK-B, Japan) and planned to be used for the upgrade of the LHC
SOME COMPLICATIONS: TRANSVERSE OFFSET (1/3)

- Luminosity in the presence of a transverse offset (but no crossing angle)

\[ x_1 = x + d_1 \]

\[ x_2 = x + d_2 \]
SOME COMPLICATIONS: TRANSVERSE OFFSET (2/3)

\[
L_{TO} = 2 M N_1 N_2 f_{rev} \int_\infty^\infty \int_\infty^\infty \int_\infty^\infty \int_\infty^\infty \rho_{1x}(x_1) \rho_{1y}(y) \rho_{1s}(s-s_0) \rho_{2x}(x_2) \rho_{2y}(y) \rho_{2s}(s+s_0) \, dx \, dy \, ds \, ds_0
\]

- Assuming \( \sigma_{1x} = \sigma_{2x} = \sigma_x \)
- \( \sigma_{1y} = \sigma_{2y} = \sigma_y \)
- \( \sigma_{1s} = \sigma_{2s} = \sigma_s \)
- \( y_1 = y_2 = y \)

Useful relations in Introduction, yields

\[
L_{TO} = L_0 F_{TO}
\]

with

\[
F_{TO} = e^{-\left(\frac{d_1-d_2}{2\sigma_x}\right)^2}
\]
SOME COMPLICATIONS: TRANSVERSE OFFSET (3/3)

\[ \frac{d_1 - d_2}{\sigma_x} \]

Transverse offset [in \( \sigma \)]
SOME COMPLICATIONS: HOURGLASS EFFECT (1/3)

- Luminosity in the presence of the Hourglass effect (and crossing angle in the x-s plane but no transverse offset)
  - Close to the IP, the beta function is given by (see course on Transverse Beam Dynamics)

\[ \beta(s) = \beta^* \left[ 1 + \left( \frac{s}{\beta^*} \right)^2 \right] \]

![Graph showing beta function and rms beam size with different beta values.]
SOME COMPLICATIONS: HOURGLASS EFFECT (2/3)

- Following the same approach as before, yields

\[
L_{CA\&\ HG} = \frac{\cos \frac{\Phi}{2} N_1 N_2 M f_{rev}}{4 \pi^{3/2} \sigma_s \sigma_x^* \sigma_y^*} \int_{-\infty}^{+\infty} ds \ e^{-s^2} \left[ \frac{\sin^2 \frac{\Phi}{2}}{2} + \frac{\cos^2 \frac{\Phi}{2}}{\sigma_s^2} \right] \left[ 1 + \left( \frac{s}{\beta^*} \right)^2 \right]
\]

\[
\Rightarrow \quad L_{CA\&\ HG} = L_{CA} \ F_{HG}
\]

with

\[
F_{HG} = \sqrt{\frac{\sin^2 \frac{\Phi}{2}}{\sigma_x^2} + \frac{\cos^2 \frac{\Phi}{2}}{\sigma_s^2}} \sqrt{\pi} \int_{-\infty}^{+\infty} ds \ e^{-s^2} \left[ \frac{\sin^2 \frac{\Phi}{2}}{2} + \frac{\cos^2 \frac{\Phi}{2}}{\sigma_s^2} \right] \left[ 1 + \left( \frac{s}{\beta^*} \right)^2 \right]
\]
Starts to become important when $\beta^*$ is comparable or smaller than the rms bunch length $\sigma_s$

LHC nominal rms bunch length $\sigma_s$: 7.5 cm

LHC nominal: 55 cm
**Integrated luminosity**

\[ L_{\text{int}} = \int_{0}^{T} L(t) \, dt \]

**Real figure of merit**

\[ L_{\text{int}} \sigma_r = \text{number of events} \]

**Let’s assume some luminosity lifetime behaviour \( \Rightarrow \) Exponential decay (due to intensity decay, emittance growth, etc.)**

\[ L(t) = L_{\text{peak}} e^{-\frac{t}{\tau_l}} \]

**What is the best run time \( t_r \)?**
Let's call $t_p$ the preparation time (time needed to put the beams in collision after the end of the previous physics fill) $=>$
Optimization of $t_r$ and $t_p$ gives the maximum luminosity

$$\langle L \rangle = \frac{1}{t_r + t_p} \int_{0}^{t_r + t_p} L(t) \, dt$$

=>

$$\langle L \rangle = L_{\text{peak}} \tau_l \frac{1 - e^{-\frac{t_r}{\tau_l}}}{t_r + t_p}$$
INTEGRATED LUMINOSITY AND MAXIMIZATION (3/4)

\[ L(t) = e^{-\frac{t}{\tau_l}} \]

\[ \tau_l = 15 \text{ h} \]

\[ \tau_p = 10 \text{ h} \]

\[ \frac{\langle L \rangle}{L_{\text{peak}}} = \tau_l \frac{1-e^{-\frac{t_r}{\tau_l}}}{t_r + t_p} \]
The average luminosity is maximum when

\[ t_r \approx \tau_l \ln \left( 1 + \sqrt{2 \frac{t_p}{\tau_l} + \frac{t_p}{\tau_l}} \right) \]

Gives \( \sim 15.5 \) h…
PILE-UP, LEVELING AND LUMINOUS REGION (1/3)

- Pile-Up (PU) = Number of events / crossing for a given luminosity

\[
PU = \frac{L \sigma_r}{M f_{rev}}
\]

- This is a limit coming from the experiments’ detectors => Better to have larger number of bunches (for the same beam intensity)

- In case the pile-up is too big, luminosity leveling techniques could be used to remain at the limit => Playing with the different parameters which can reduce the luminosity (transverse beam offset, \( \beta^* \), etc.)
PILE-UP, LEVELING AND LUMINOUS REGION (2/3)

Luminous region

In the case of crossing angle only

\[
L_{CA}(s_{\text{max}}) = \int_{-s_{\text{max}}}^{+s_{\text{max}}} ds e^{\frac{-s^2}{2}} \left( 1 + \left( \frac{s}{\beta^*} \right)^2 \right)^{\sigma_x^2} \left[ 1 + \left( \frac{s}{\beta^*} \right)^2 \right] \]

\[
+ \int_{-\infty}^{+\infty} ds e^{\frac{-s^2}{2}} \left( 1 + \left( \frac{s}{\beta^*} \right)^2 \right)^{\sigma_y^2} \left[ 1 + \left( \frac{s}{\beta^*} \right)^2 \right] \]

W. Herr
90% of total luminosity for $s_{\text{max}} = 7.4 \text{ cm}$
SUMMARY: HOW TO REACH HIGH LUMINOSITY?

- High beam intensities
  - High bunch intensity => More efficient (for the same beam intensity) but pile-up issue for the experiments’ detectors
  - High number of bunches => Less efficient but better for the pile-up
- Small transverse beam sizes (small transverse emittance and beta function at the IP)
- High energy
- Small crossing angle
- Small transverse offset
- Short bunches