Correction of the JUAS exam on longitudinal beam dynamics

Note for the equation:

"np.sqrt" is the square root

"np.pi" is the number π

1) a.

```
import numpy as np
m_p=1.67e-27
e=1.6e-19
c=3e8

#question 1.a.
B=0.143
Ekin=3e9
Etot=Ekin*e+m_p*c**2
gamma=Etot/(m_p*c**2)
beta=np.sqrt(1-1/gamma**2)
p_0=Etot/c*beta
bending_radius=p_0/(e*B)
print bending_radius
```

```
89.1779772368
```

The bending radius is 89.2 m.

```
magnetic_length=bending_radius*2*np.pi/96
print magnetic_length
```

```
5.83668496144
```

The magnetic length of a dipole is 5.84 m.

```
circumference=1567.5
print bending_radius*2*np.pi/circumference
```

```
0.357462045485
```

35.7 % of the ring is made of dipoles.

Note: In the absence of other information, the length of the dipole is considered to be equal to its magnetic length. A dipole is in fact always slightly longer than its magnetic length.

1) b.

If the magnetic length does not change at top energy, then the same bending radius is needed from all dipoles and the ratio *"p_0/B = ρ/e"* is constant. Therefore

```
# question 1)b.
#at injection energy
p_0_inj=p_0
B_inj=B
# at top energy
Ekin=50e9
Etot=Ekin*e+m_p*c**2
gamma=Etot/(m_p*c**2)
beta=np.sqrt(1-1/gamma**2)
p_0_top=Etot/c*beta
B_top=p_0_top*B_inj/p_0_inj
print B_top
```

```
1.90371039712
```

And the magnetic field required at top energy is 1.9 T.

1) c.

from the relativistic beta computed above at injection and top energy, we get:

```
# question 1) c.
# recomputing beta for injection
Ekin=3e9
Etot=Ekin*e+m_p*c**2
gamma=Etot/(m_p*c**2)
beta=np.sqrt(1-1/gamma**2)
f_inj=beta*c/circumference
print f_inj
```

```
185866.572462
```

The revolution frequency at injection is 186 kHz

```
# recomputing beta for top energy
Ekin=50e9
Etot=Ekin*e+m_p*c**2
gamma=Etot/(m_p*c**2)
beta=np.sqrt(1-1/gamma**2)
f_top=beta*c/circumference
print f_top
```

```
191355.014274
```

The revolution frequency at top energy is 191 kHz

```
# question 1) d.
f_top -f_inj
```

```
5488.4418123867363
```

And the revolution frequency change between injection and top energy is 5.49 kHz.

1) d.

```
harmonic=9
f_rf_inj=harmonic*f_inj
print f_rf_inj
```

```
1672799.15216
```

The RF frequency at injection energy is 1.67 MHz.

```
harmonic=9
f_rf_top=harmonic*f_top
print f_rf_top
```

```
1722195.12847
```

The RF frequency at top energy is 1.72 MHz

2) a.

See slide 44 in the course:

Increasing the energy of a particle circulating in a synchrotron changes both its velocity and the length of its trajectory. Both velocity and trajectory length affect revolution frequency. Transition energy is defined (for a positive momentum compaction factor) as the energy for which the velocity variation is compensated by the trajectory variation.

- Below transition energy an increase of energy leads to higher revolution frequency → the velocity variation dominates the trajectory length variation
- Above transition energy an increase of energy leads to lower revolution frequency → the trajectory length variation dominates the velocity variation

2) b.

```
gamma_tr=22.4
alpha_p= 1/gamma_tr**2
print alpha_p
```

```
0.00199298469388
```

The momentum compaction factor is $1.99\ 10^{-3}$.

At injection:

```
Ekin=3e9
Etot=Ekin*e+m_p*c**2
gamma=Etot/(m_p*c**2)
eta=1/gamma**2-1/gamma_tr**2
print eta
```

```
0.0548691745055
```

The slippage factor at injection is 0.0549.

At top energy:

```
Ekin=50e9
Etot=Ekin*e+m_p*c**2
gamma=Etot/(m_p*c**2)
eta=1/gamma**2-1/gamma_tr**2
print eta
```

```
-0.00165291277521
```

The slippage factor at top energy is -1.65 10^{-3}.

The beam indeed crosses transition during the cycle: the slippage factor changes sign between injection and top energy. It can of course also be seen from gamma or energy.

2) c.

When crossing transition, the synchronous phase ϕ_s needs to be changed from ϕ_s to π-ϕ_s , as otherwise the longitudinal motion of the synchronous particle would be unstable.

3) a.

```
Vrf=80e3
Ekin=3e9
Etot=Ekin*e+m_p*c**2
gamma=Etot/(m_p*c**2)
beta=np.sqrt(1-1/gamma**2)
eta=1/gamma**2-1/gamma_tr**2
Qs=np.sqrt(e*Vrf*harmonic*eta/(2*np.pi*beta**2*Etot))
print Qs
```

```
0.00130088612004
```

At injection energy, the synchrotron tune Qs is 1.3 10^{-3} .

```
print 1/Qs
```

```
768.706795001
```

The particles with small amplitudes in the longitudinal phase space take 769 machine turns to perform one turn in the longitudinal phase space. The synchrotron tune decreases with amplitude in the longitudinal phase space. It can only be defined for stable particles (i.e. inside the separatrix). The synchrotron tune tends towards 0 when the particle tends towards the separatrix: particles move very slowly in phase space.

3) b.

To draw the bucket in the longitudinal phase space, one needs the synchronous phase and the maximum of the bucket in terms of ΔE and phase.

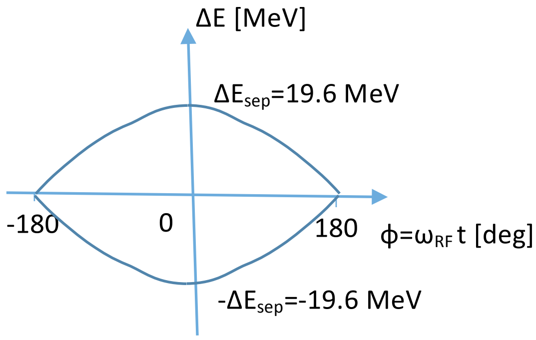
At injection energy below transition with no acceleration, the energy gain per turn should be 0 and the synchronous particle should be stable. From these two constraints, we get that the synchronous phase should be 0 degree (see slide 63). The bucket extends from -180 to 180 degrees.

The maximum acceptance ΔE_{sep} is given in slide 85 of the course with $G(\phi_s)=1$ for $\phi_s=0$.

```
Vrf=80e3
Bdot=0
Ekin=3e9
Etot=Ekin*e+m_p*c**2
gamma=Etot/(m_p*c**2)
beta=np.sqrt(1-1/gamma**2)
eta=1/gamma**2-1/gamma_tr**2
phi_s=np.arcsin(2*np.pi*bending_radius*circumference/(2*np.pi)*Bdot/Vrf)
print phi_s
G=np.sqrt(2*np.cos(phi_s)-(np.pi-2*phi_s)*np.sin(phi_s))/np.sqrt(2)
print G
deltaEsep=np.sqrt(2*beta**2*Etot*e*Vrf/(np.pi*harmonic*eta))
print deltaEsep/e/1e6
```

```
0.0
1.0
19.574957794
```

The maximum acceptance is ΔE_{sep}=19.6 MeV.



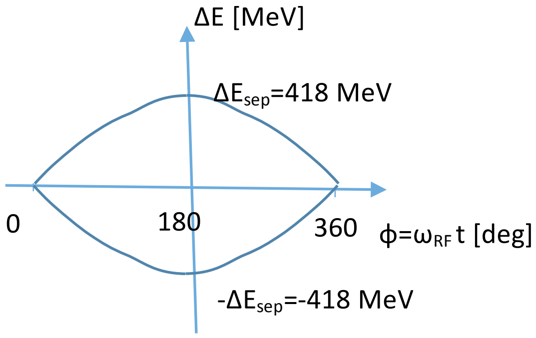
3) c.

At top energy without acceleration, the synchronous phase needs to change to π to be stable. The bucket now extends from 0 to 360 degrees. The bucket now extends from 0 to 360 degrees.

```
Vrf=80e3
Bdot=0
Ekin=50e9
Etot=Ekin*e+m_p*c**2
gamma=Etot/(m_p*c**2)
beta=np.sqrt(1-1/gamma**2)
eta=1/gamma**2-1/gamma_tr**2
phi_s=np.arcsin(2*np.pi*bending_radius*circumference/(2*np.pi)*Bdot/Vrf)
print phi_s
G=np.sqrt(2*np.cos(phi_s)-(np.pi-2*phi_s)*np.sin(phi_s))/np.sqrt(2)
print G
deltaEsep=np.sqrt(2*beta**2*Etot*e*Vrf/(np.pi*harmonic*-eta))
print deltaEsep/e/1e6
```

```
0.0
1.0
417.533936396
```

The maximum acceptance ΔE_{sep} is now 418 MeV.

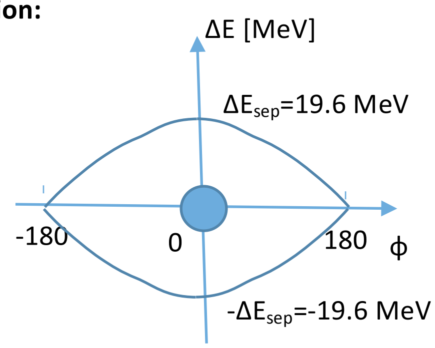


There is much more acceptance at top energy in absolute terms (more than a factor 20), but only 65% more when looking in relative terms (ΔE/Etot).

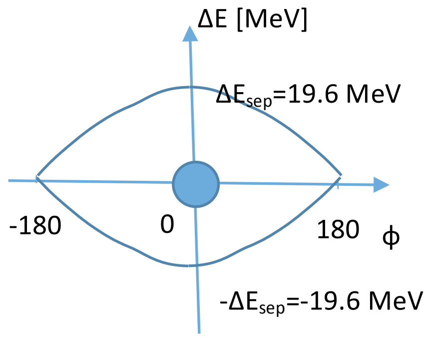
3) d.

- if the bunch is well matched, there is no filamentation and no emittance blow up:

at injection:

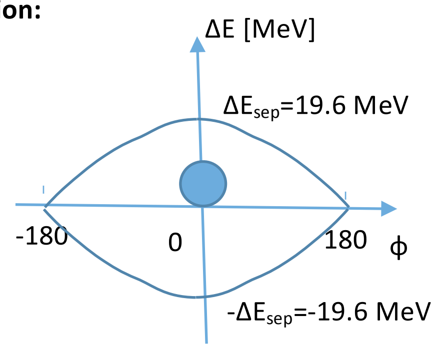


a long time after injection:

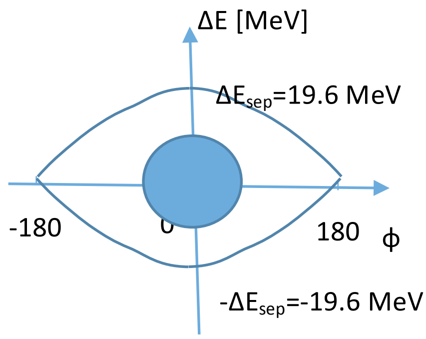


- if there is an injection error of +5 MeV, then there is filamentation and emittance blow up:

at injection:

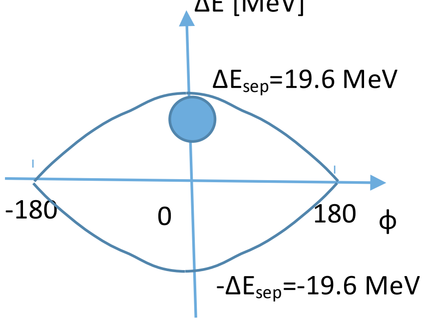


a long time after injection:

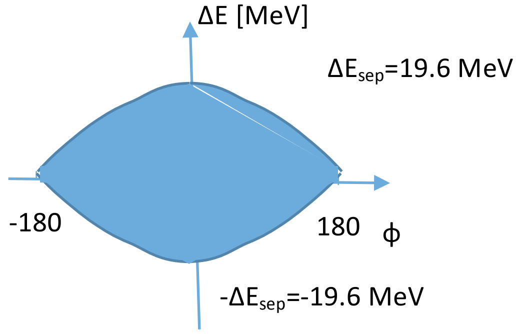


- the maximum allowed energy error at injection would be of the order of 19.6 - 5 =14.6 MeV, to avoid losses from the bucket:

at injection:



a long time after injection:



4) a.

The lowest voltage is given by the synchronous phase equation in slide 63: the argument of the "arcsin" function should be lower than 1 to have a solution. This means $\hat{V}_{RF} \geq 2\pi R\rho\dot{B}$ with the notations of the course.

The magnetic field rate $\dot{B}$ can be obtained from the length of the ramp and the magnetic fields at injection and top energy, assuming the increase is linear with time.

```
Bdot=(1.9-0.143)/1.9
print Bdot
Vrf_min=circumference*bending_radius*Bdot
print Vrf_min
```

```
0.924736842105
129265.707454
```

The lowest effective voltage to have a stable area is therefore 129 kV. The voltage needs to be higher to get substantial acceptance though.

4) b.

The transit time factor is defined in slide 35:

```
Ekin=3e9
Etot=Ekin*e+m_p*c**2
gamma=Etot/(m_p*c**2)
beta=np.sqrt(1-1/gamma**2)
frf=beta*c/circumference*harmonic
gap=0.035
transitT=np.sin(2*np.pi*frf*gap/(2*beta*c))/(2*np.pi*frf*gap/(2*beta*c))
print transitT
```

```
0.999999933571
```

In our case, the transit time factor is very close to 1.

Looking at the formula, to first order, the transit time factor does not change with momentum.

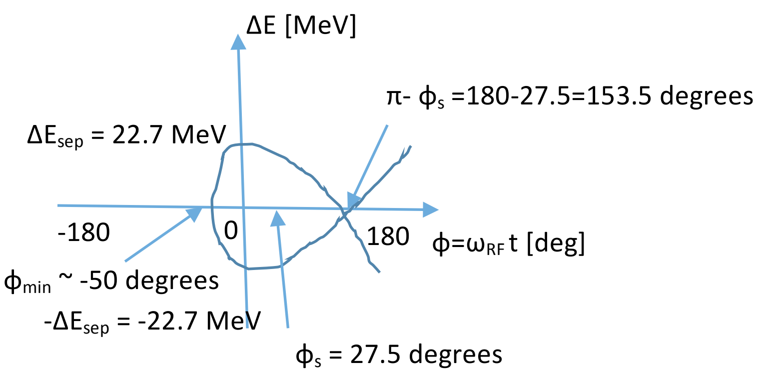
4) c.

Using the same equations to get the synchronous phase and bucket acceptance as in question 3)b. we get close to injection:

```
Vrf=280e3
Bdot=(1.9-0.143)/1.9
Ekin=3e9
Etot=Ekin*e+m_p*c**2
gamma=Etot/(m_p*c**2)
beta=np.sqrt(1-1/gamma**2)
eta=1/gamma**2-1/gamma_tr**2
phi_s=np.arcsin(2*np.pi*bending_radius*circumference/(2*np.pi)*Bdot/Vrf)
print phi_s*180/np.pi
G=np.sqrt(2*np.cos(phi_s)-(np.pi-2*phi_s)*np.sin(phi_s))/np.sqrt(2)
print G
deltaEsep=np.sqrt(2*beta**2*Etot*e*Vrf/(np.pi*harmonic*eta))*G
print deltaEsep/e/1e6
```

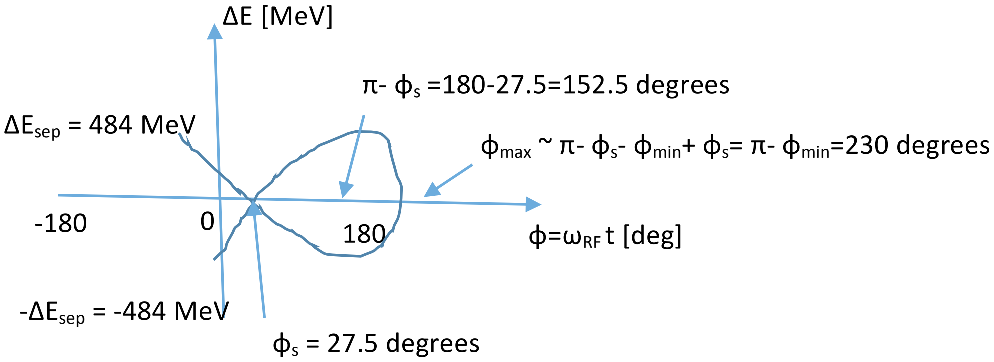
```
27.4944855786
0.619204621234
22.6761356039
```

The stable phase is the synchronous phase (27.5 degrees) and ΔE_{sep} is 22.7 MeV. The minimum phase can be estimated from slide 84.



Close to top energy, we have:

The stable phase is π- ϕ_s (152.5degrees) and ΔE_{sep} is 484 MeV. The maximum phase can be found from the fact that the phase acceptance does not change from the case close to injection energy.



5) a.

For α_p = - 0.001, gamma transition is a pure imaginary number : j31.6.

```
alpha_p=-0.001
im_gamma_tr=np.sqrt(1/-alpha_p)
print im_gamma_tr

Ekin=3e9
Etot=Ekin*e+m_p*c**2
gamma=Etot/(m_p*c**2)
eta_inj=1/gamma**2-alpha_p

Ekin=50e9
Etot=Ekin*e+m_p*c**2
gamma=Etot/(m_p*c**2)
eta_top=1/gamma**2-alpha_p
print eta_inj, eta_top
```

```
31.6227766017
0.0578621591994 0.00134007191867
```

The slippage factor at injection and top energy are now of the same sign (0.058 at injection and 0.00134 at top energy). The transition energy is therefore not crossed and the slippage factor is such that the energy is always below transition (i.e. the trajectory length variation adds up to the velocity variation in the frequency dependence on energy).

5) b.

Designing a machine with negative momentum compaction factor allows avoiding all the issues related with transition crossing, longitudinal phase change but also many more issues that Elias mentioned quickly during the course (e.g. instabilities around transition).

5) c.

The usual equations apply and we get at injection:

```
Vrf=80e3
Bdot=0
Ekin=3e9
Etot=Ekin*e+m_p*c**2
gamma=Etot/(m_p*c**2)
beta=np.sqrt(1-1/gamma**2)
alpha_p=-0.001
eta=1/gamma**2-alpha_p
phi_s=np.arcsin(2*np.pi*bending_radius*circumference/(2*np.pi)*Bdot/Vrf)
print phi_s*180/np.pi
G=np.sqrt(2*np.cos(phi_s)-(np.pi-2*phi_s)*np.sin(phi_s))/np.sqrt(2)
print G
deltaEsep=np.sqrt(2*beta**2*Etot*e*Vrf/(np.pi*harmonic*eta))*G
print deltaEsep/e/1e6
Qs=np.sqrt(e*Vrf*harmonic*eta/(2*np.pi*beta**2*Etot))
print Qs
```

```
0.0
1.0
19.0619677127
0.00133589518556
```

Qs= 1.33 10^{-3}, and the bucket acceptance is from -180 to 180 degrees and -19.1 MeV to 19.1 MeV.

At top energy:

```
Vrf=80e3
Bdot=0
Ekin=50e9
Etot=Ekin*e+m_p*c**2
gamma=Etot/(m_p*c**2)
beta=np.sqrt(1-1/gamma**2)
alpha_p=-0.001
eta=1/gamma**2-alpha_p
phi_s=np.arcsin(2*np.pi*bending_radius*circumference/(2*np.pi)*Bdot/Vrf)
print phi_s*180/np.pi
G=np.sqrt(2*np.cos(phi_s)-(np.pi-2*phi_s)*np.sin(phi_s))/np.sqrt(2)
print G
deltaEsep=np.sqrt(2*beta**2*Etot*e*Vrf/(np.pi*harmonic*eta))*G
print deltaEsep/e/1e6
Qs=np.sqrt(e*Vrf*harmonic*eta/(2*np.pi*beta**2*Etot))
print Qs
```

```
0.0
1.0
463.716655348
5.49145487897e-05
```

Qs= 5.49 10^{-5}, and the bucket acceptance is from -180 to 180 degrees and -464 MeV to 464 MeV.