LONGITUDINAL BEAM DYNAMICS

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BE Department - CERN

The present transparencies are inherited from Frank Tecker (CERN-BE), who gave this course two years ago (I already gave this course in 2011&12) and who inherited them from Roberto Corsini (CERN-BE), who gave this course in the previous years, based on the one written by Louis Rinolfi (CERN-BE) who held the course at JUAS from 1994 to 2002 (see CERN/PS 2000-008 (LP)).


Material from Joel LeDuff’s Course at the CERN Accelerator School held at Jyvaskyla, Finland the 7-18 September 1992 (CERN 94-01) has been used as well:


I attended the course given by Louis Rinolfi in 1996 and was his assistant in 2000 and 2001 (and the assistant of Michel Martini for his course on transverse beam dynamics)

This course and related exercises (as well as other courses) can be found in my web page:

http://emetral.web.cern.ch/emetral/

8 Lectures
4 Tutorials

Fields & Forces
Relativity
Acceleration (electrostatic, RF)
Synchrotons
Longitudinal phase space
Momentum Compaction
Transition energy
Synchrotron oscillations

Examination: WE 06/02/2013
(09:00 to 10:30)

LESSON I

Fields & Forces

Acceleration by time-varying fields

Relativistic equations
Fields and force

Equation of motion for a particle of charge $q$

$$\vec{F} = \frac{\text{d}\vec{p}}{\text{d}t} = q \left( \vec{E} + \vec{v} \times \vec{B} \right)$$

- $\vec{p} = m \vec{v}$: Momentum
- $\vec{v}$: Velocity
- $\vec{E}$: Electric field
- $\vec{B}$: Magnetic field

The fields must satisfy Maxwell's equations

The integral forms, in vacuum, are recalled below:

1. Gauss's law (electrostatic)
   $$\int_{\mathcal{V}} \vec{E} \cdot \text{d}\vec{s} = \frac{1}{\varepsilon_0} \int_{\mathcal{V}} \rho \, \text{d}\mathcal{V}$$

2. No free magnetic poles (magnetostatic)
   $$\int_{\mathcal{V}} \vec{B} \cdot \text{d}\vec{s} = 0$$

3. Ampere's law (modified by Gauss) (magnetostatic)
   $$\int_{\mathcal{V}} \vec{B} \cdot \text{d}\vec{l} = \mu_0 \int_{\mathcal{V}} \vec{j} \cdot \text{d}\vec{V} + \frac{1}{c^2} \int_{\mathcal{V}} \frac{\partial \vec{E}}{\partial t} \cdot \text{d}\vec{s}$$

4. Faraday's law (magnetic varying)
   $$\int_{\mathcal{V}} \vec{E} \cdot \text{d}\vec{l} = -\int_{\mathcal{V}} \frac{\partial \vec{B}}{\partial t} \cdot \text{d}\vec{s}$$

Maxwell's equations

The differential forms, in vacuum, are recalled below:

1. Gauss's law
   $$\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho(\vec{r}, t)$$

2. No free magnetic poles
   $$\nabla \cdot \vec{B} = 0$$

3. Ampere's law (modified by Gauss)
   $$\nabla \times \vec{B} = \mu_0 \mu_r \vec{j}(\vec{r}, t) + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

4. Faraday's law
   $$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Constant electric field

- Direction of the force always parallel to the field
- Trajectory can be modified, velocity also ⇒ momentum and energy can be modified

This force can be used to accelerate and decelerate particles
This force cannot modify the energy magnetic rigidity:

$$\omega = 2\pi f = \frac{e}{m} B$$

1. Direction always perpendicular to the velocity
2. Trajectory can be modified, but not the velocity

$$e v B = \frac{m v^2}{\rho}$$

magnetic rigidity: $$B \rho = \frac{p}{e}$$ angular frequency: $$\omega = 2\pi f = \frac{e}{m} B$$

Application: spectrometer

Important relationship:

$$B \rho = \frac{p}{e} \Rightarrow \rho = \frac{p}{e B}$$

Practical units:

$$B \rho \text{[Tm]} = \frac{p \text{[GeV/c]}}{0.3}$$

Real life example: CTF3

Time-resolved spectrum

Larmor formula

An accelerating charge radiates a power $$P$$ given by:

$$P = \frac{2}{3} \frac{r_e}{m_e c} \left\{ \beta^2 + \gamma^2 \rho^2 \right\}$$

Energy lost on a trajectory $$L$$

$$W = \int L \left( \frac{P}{v} \right) ds$$

For electrons in a constant magnetic field

$$W \left[ \text{eV/turn} \right] = 88 \cdot 10^7 \frac{E \left[ \text{GeV} \right]}{\rho \left[ \text{m} \right]}$$

Comparison of magnetic and electric forces

$$|\vec{B}| = 1 \text{ T}$$

$$|\vec{E}| = 10 \text{ MV/m}$$

$$\frac{F_{\text{MACN}}}{F_{\text{ELC}}} = \frac{e v B}{e E} = \beta c \frac{B}{E} = 3 \cdot 10^4 \frac{1}{10} \beta = 30 \beta$$
**Acceleration by time-varying magnetic field**

A variable magnetic field produces an electric field (Faraday’s Law):

\[
\int \mathbf{E} \cdot d\mathbf{l} = -\int \frac{d\mathbf{B}}{dt} \cdot d\mathbf{s} = -\frac{d\Phi}{dt}
\]

It is the Betatron concept.

The varying magnetic field is used to guide particles on a circular trajectory as well as for acceleration.

**Betatron**

\[
\int \mathbf{E} \cdot d\mathbf{l} = 2\pi R E = -\pi R \frac{d\Phi}{dt} - \frac{d\mathbf{B}_m}{dt}
\]

\[
\frac{dp}{dt} = eE = \frac{1}{2} eR \frac{d\mathbf{B}_m}{dt}
\]

\[
B_f = \frac{1}{2} B_m + \text{const.}
\]

**Acceleration by time-varying electric field**

- Let \( V_{RF} \) be the amplitude of the RF voltage across the gap.
- The particle crosses the gap at a distance \( r \).
- The energy gain is:

\[
\Delta E = e \int_{s_1}^{s_2} E(r, t) d\bar{s}
\]

[MeV] [n] [MV/m] (1 for electrons or protons)

In the cavity gap, the electric field is supposed to be:

\[ E(s, r, t) = E_0(s, r) \cdot E_1(t) \]

In general, \( E_0(t) \) is a sinusoidal time variation with angular frequency \( \omega_{RF} \):

\[ E_1(t) = E_s \sin(\omega_{RF} t) \quad \text{where} \quad \Phi(t) = \int_{0}^{t} \omega_{RF} dt + \Phi_2 \]

**Convention**

1. For circular accelerators, the origin of time is taken at the zero crossing of the RF voltage with positive slope.
2. For linear accelerators, the origin of time is taken at the positive crest of the RF voltage.

Time \( t_0 \) chosen such that:

\[
E_1(t) = E_s \sin(\omega_{RF} t) \quad E_2(t) = E_s \cos(\omega_{RF} t)
\]
Relativistic Equations

\[ E = mc^2 \]

- \( \beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \)
- \( E = E_{\text{kin}} + E_0 \)
- \( p = mv = \frac{E}{c} = \beta \gamma m_0 c \)

<table>
<thead>
<tr>
<th>Energy</th>
<th>Momentum</th>
<th>Mass</th>
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<td>( eV )</td>
<td>( eV/c )</td>
<td>( eV/c^2 )</td>
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</table>

\[ p c^2 = E^2 - E_0^2 \]
\[ \gamma = 1 + \frac{E_{\text{kin}}}{E_0} \]
\[ \rho \left[ GeV/c \right] = 0.3 \left[ B \right] \rho \left[ m \right] \]

First derivatives

\[ \frac{d\beta}{\beta} = \beta \gamma^2 \gamma^2 \frac{dy}{y} \]
\[ \frac{dE}{E} = \gamma \frac{dE_{\text{kin}}}{E_{\text{kin}}} \]
\[ \frac{dy}{y} = \left( \gamma^2 - 1 \right) \frac{d\beta}{\beta} \]

Logarithmic derivatives

\[ \frac{dE}{E} = \left( \beta \gamma \right)^2 \frac{dy}{y} \]
\[ \frac{dp}{p} = \frac{dE}{E} - \frac{1}{y} \frac{dE_{\text{kin}}}{E_{\text{kin}}} \]

LESSON II

An overview of particle acceleration

- Transit time factor
- Main RF parameters
- Momentum compaction
- Transition energy

Relativistic Equations

\[ E = mc^2 \]

- \( \beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \)
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\[ p c^2 = E^2 - E_0^2 \]
\[ \gamma = 1 + \frac{E_{\text{kin}}}{E_0} \]
\[ \rho \left[ GeV/c \right] = 0.3 \left[ B \right] \rho \left[ m \right] \]
Electrostatic accelerators

- The potential difference between two electrodes is used to accelerate particles.
- Limited in energy by the maximum high voltage (~ 10 MV).
- Present applications: x-ray tubes, low energy ions, electron sources (thermosonic guns).

Electric field potential and beam trajectories inside an electron gun (LEP Injector Linac at CERN), computed with the code E-GUN.

Electrostatic accelerator

Protons & Ions

750 kV Cockroft-Walton source of LINAC 2 (CERN) © CERN Geneva

Synchronism condition

\[ g \ll L \]

\[ L = \beta \cdot T_{RF} = \beta \cdot \lambda_{RF} \]

\[ \omega_{RF} = \frac{2\pi \cdot V}{L} \]
**Proton and ion linacs**

(Alvarez structure)

**LINAC 2 (CERN)**

**LINAC 1 (CERN)**

© CERN Geneva

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**Electron Linac**

Electrons are light \( \Rightarrow \) fast acceleration

\( \Rightarrow \) \( \beta = 1 \) already at an energy of a few MeV

Uniform disk-loaded waveguide, travelling wave

(up to 50 GeV, \( f \sim 3 \text{ GHz} \) - S-band)

\[ E(\tau, t) = E_0 \, e^{i(\omega t - \frac{2\pi}{\lambda} \tau)} \]

**Wave number** \( k = \frac{2\pi}{\lambda_{RF}} \)

**Phase velocity** \( v_{ph} = \frac{\omega}{k} \)

**Group velocity** \( v_g = \frac{\partial \omega}{\partial k} \)

**Synchronism condition**

\[ \gamma = \frac{v_g}{c} = \frac{\omega}{k} = v_{ph} \]

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**Electron Linacs & structures**

**LEP Injector Linac (LIL)**

**CLIC Accelerating Structures (30 GHz & 11 GHz)**

© CERN Geneva

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**Cyclotron**

**Synchronism condition**

\[ \omega = \omega_{RF}, \quad 2\pi \beta = v_g T_{RF} \]

**Cyclotron frequency**

\[ \omega = \frac{q B}{m \gamma} \]

1. \( \gamma \) increases with the energy 
   \( \Rightarrow \) no exact synchronism

2. if \( v \ll c \Rightarrow \gamma = 1 \)
**Cyclotron**

- **TRIUMF 520 MeV cyclotron**
- **University of British Columbia - Canada**

**Cyclotron (H+ accelerated, protons extracted)**

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**Synchrocyclotron**

Same as cyclotron, except a modulation of \( \omega_{RF} \)

- \( B = \text{constant} \)
- \( \gamma \omega_{RF} = \text{constant} \)
- \( \omega_{RF} \) decreases with time

The condition:

\[
\omega(t) = \omega_{RF}(t) = \frac{q B}{m_0 \gamma(t)}
\]

Allows to go beyond the non-relativistic energies

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**Synchrotron**

- \( \omega_{RF} \) and \( \omega \) increase with energy
- To keep particles on the closed orbit, \( B \) should increase with time

**Synchronism condition**

\[
\frac{2\pi R}{\nu_j} = h T_{RF}
\]

- \( T_j = h T_{RF} \)
- \( h \) integer, harmonic number

---

**Synchrotron**

- In reality, the orbit in a synchrotron is not a circle, straight sections are added for RF cavities, injection and extraction, etc...
- Usually the beam is pre-accelerated in a linac (or a smaller synchrotron) before injection
- The bending radius \( \rho \) does not coincide to the machine radius \( R = L/2\pi \)
Parameters for circular accelerators

The basic principles for the common circular accelerators are based on the two relations:

1. The Lorentz equation: The orbit radius can be expressed as:
   \[ R = \frac{T \frac{e}{m}}{B} \]

2. The synchronicity condition: The revolution frequency can be expressed as:
   \[ f = \frac{e B}{2 \pi m} \]

According to the parameter we want to keep constant or let vary, one has different acceleration principles. They are summarized in the table below:

<table>
<thead>
<tr>
<th>Machine</th>
<th>Energy ((\gamma))</th>
<th>Velocity</th>
<th>Field</th>
<th>Orbit</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cyclotron</td>
<td>(\approx 1)</td>
<td>var.</td>
<td>const.</td>
<td>~(v)</td>
<td>const.</td>
</tr>
<tr>
<td>Synchrocyclotron</td>
<td>var.</td>
<td>var.</td>
<td>(B(r)/\gamma(t))</td>
<td>~(p)</td>
<td>~(R)</td>
</tr>
<tr>
<td>Proton/Ion synchrotron</td>
<td>var.</td>
<td>var.</td>
<td>~(p)</td>
<td>(R)</td>
<td>~(v)</td>
</tr>
<tr>
<td>Electron synchrotron</td>
<td>var.</td>
<td>const.</td>
<td>~(p)</td>
<td>(R)</td>
<td>const.</td>
</tr>
</tbody>
</table>

Transit time factor

RF acceleration in a gap \(g\)

\[ E(s, r, t) = E_i(s, r) \cdot E_j(t) \]

\[ E_i(s, r) = \frac{V_{sp}}{g} \]

\[ E_j(t) = \sin(\omega_{sp} t + \phi_i) \]

\[ \Delta E = e \int_{-g/2}^{g/2} E(s, r, t) \, ds \]

\[ \Delta E = e V_{sp} T_s \sin \phi_i \]

\[ T_s = \frac{\sin \frac{\omega_{sp} g}{2v}}{\frac{\omega_{sp} g}{2v}} \]

\(T_s\) is called transit time factor

- \(T_s < 1\)
- \(T_s \rightarrow 1\) if \(g \rightarrow 0\)

Transit time factor II

In the general case, the transit time factor is given by:

\[ T_a = \frac{\int_{-g}^{g} E_i(s, r) \cos \left( \frac{\omega_{sp} s}{V} \right) \, ds}{\int_{-g}^{g} E_i(s, r) \, ds} \]

It is the ratio of the peak energy gained by a particle with velocity \(v\) to the peak energy gained by a particle with infinite velocity.
### Main RF parameters

In order to accelerate particles, longitudinal fields must be generated in the direction of the desired acceleration.

\[
E(x,t) = E_x(x) \cdot E_z(t) = E_x \sin\left(\omega_{RF} \cdot dt + \phi_0\right)
\]

\[
\omega_{RF} = 2 \pi f_{RF}
\]

\[
\Delta E = e V_{RF} T_s \sin \phi_0
\]

Such electric fields are generated in RF cavities characterized by the voltage amplitude, the frequency and the phase.

### Harmonic number

- \(T_{rev} = h T_{RF} \Rightarrow f_{RF} = h f_{rev}\)

<table>
<thead>
<tr>
<th>(h)</th>
<th>(f_{rev})</th>
<th>(f_{RF})</th>
</tr>
</thead>
<tbody>
<tr>
<td>AA</td>
<td>EPA</td>
<td>PS</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>20</td>
</tr>
</tbody>
</table>

### Dispersion

- Nominal trajectory
- Reference = design = nominal trajectory
- Closed orbit (circular machine)

\[
x(s) = D_x(s) \frac{\Delta p}{p}
\]

### Example: constant magnetic field

\[
d_s = \rho \, d\theta
\]

\[
d_s = (\rho + x) d\theta
\]

\[
\frac{ds}{d\theta} = \frac{\rho + x}{\rho} d\theta
\]

\[
\alpha_s = \frac{1}{L} \int D_x(s) \, ds
\]

Therefore, for small momentum deviation, to first order it is:

\[
\frac{\Delta L}{L} = \alpha_p \frac{\Delta p}{p}
\]

To first order, only the bending magnets contribute to a change of the trajectory length if \(r = \infty\) in the straight sections.
### Longitudinal phase space

- **Δp/p**
- **Acceleration:** move forward
- **Deceleration:** move backward

The particle trajectory in the phase space (Δp/p, φ) describes its longitudinal motion.

**Emittance:** phase space area including all the particles.

NB: if the emittance contour corresponds to a possible orbit in phase space, its shape does not change with time (matched beam).

### Bunch compressor

- **Δp/p**
- **Acceleration:** move forward
- **Deceleration:** move backward

The particle trajectory in the phase space (Δp/p, φ) describes its longitudinal motion.

**Bunch compression**

**Bunch compressor**

**Longitudinal phase space evolution for a bunch compressor (PARMELA code simulations)**

**Momentum compaction in a ring**

In a circular accelerator, a nominal closed orbit is defined for the nominal momentum p.

For a particle with a momentum deviation Δp produces an orbit length variation ΔC with:

\[
\frac{\Delta C}{C} = \alpha_p \frac{\Delta p}{p}
\]

**C** = circumference

\(\alpha_p\) is the momentum compaction factor defined by the ratio:

\[
\alpha_p = \frac{dC}{C} \frac{dR}{R} \frac{dp}{dp}
\]

**N.B.** in most circular machines, **α_p** is positive: higher momentum means longer circumference.
Momentum compaction as a function of energy

\[ E = \frac{pc}{\beta} \implies \frac{dE}{E} = \beta \frac{dp}{p} \]

\[ \alpha_f = \beta^2 \frac{E}{R} \frac{dR}{dE} \]

Momentum compaction as a function of magnetic field

Definition of average magnetic field

\[ <B> = \frac{1}{2\pi R_c} \int B_s \, ds = \frac{1}{2\pi R_c} \left( \int B_s \, ds \right) \]

\[ <B> = \frac{B_f \rho}{R} \]

\[ \frac{d<B>}{dB_f} = \frac{d(\frac{B_f \rho}{R})}{dB_f} = \frac{\frac{d\rho}{R} - \frac{dR}{R}}{\frac{B_f \rho}{R}} \]

\[ B_f \rho = \frac{p}{e} \]

\[ \frac{d<B>}{dR} = \frac{d\frac{B_f \rho}{R}}{dR} = \frac{d(\frac{B_f \rho}{R})}{dR} = \frac{\frac{d\rho}{R} - \frac{dR}{R}}{\frac{B_f \rho}{R}} \]

For \( B_f = \text{const.} \)

\[ \alpha_f = 1 - \frac{\frac{d<B>}{dp}}{\frac{d<B>}{p}} \]

Transition energy

Proton (ion) circular machine with \( \alpha_f \) positive

1. Momentum larger than the nominal \( p + \Delta p \) \( \implies \) longer orbit \( (C + \Delta C) \)
2. Momentum larger than the nominal \( p + \Delta p \) \( \implies \) higher velocity \( (v + \Delta v) \)

What happens to the revolution frequency \( f = \frac{v}{C} \)?

- At low energy, \( v \) increases faster than \( C \) with momentum
- At high energy \( v \approx c \) and remains almost constant

There is an energy for which the velocity variation is compensated by the trajectory variation \( \implies \) transition energy

Transition energy - quantitative approach

We define a parameter \( \eta \) (revolution frequency spread per unit of momentum spread):

\[ \eta = \frac{df}{f} \frac{\partial \alpha}{\partial p} - \frac{\partial \alpha}{\partial p} \]

\[ f = \frac{v}{C} \implies \frac{df}{f} = \frac{\partial \beta}{\partial \beta} \frac{dC}{C} \]

\[ \eta = \frac{df}{f} \frac{\partial \beta}{\partial \beta} \frac{dC}{C} \quad \text{from} \quad p = \frac{m \gamma c \beta}{\sqrt{1 - \beta^2}} \]

\[ \frac{d\beta}{\beta} = \frac{1}{\gamma^2} \frac{dp}{p} \quad \text{definition of momentum compaction factor:} \]

\[ \frac{dC}{C} = \alpha_f \frac{dp}{p} \]

\[ \frac{df}{f} = \left( 1 - \alpha_f \right) \frac{dp}{p} \]
Transition energy – quantitative approach

The transition energy is the energy that corresponds to $\eta = 0$ ( $\alpha_p$ is fixed, and $\gamma$ variable )

$$\gamma_r = \frac{1}{\alpha_p}$$

The parameter $\eta$ can also be written as

- At low energy $\eta > 0$
- At high energy $\eta < 0$

N.B.: for electrons, $\gamma \gg \gamma_{tr} \Rightarrow \eta < 0$ for linacs $\alpha_p = 0 \Rightarrow \eta > 0$

LESSON III

Equations related to synchrotrons

Synchronous particle

Synchrotron oscillations

Principle of phase stability

Equations related to synchrotrons

- Momentum
- Orbit radius
- Magnetic field
- Rev. frequency
- Transition energy

\[
\frac{dp}{p} = \gamma_r \frac{dB}{B} + \frac{dR}{R} + \frac{df}{f} \\
\frac{dp}{p} = \gamma_r \frac{dB}{B} + \frac{dR}{R} \\
\frac{dB}{B} = \gamma_r \frac{df}{f} + \left( \gamma^2 - \gamma_r^2 \right) \frac{dR}{R}
\]

I - Constant radius

Beam maintained on the same orbit when energy varies

\[
\frac{dp}{p} = \frac{dB}{B} \\
\frac{dp}{p} = \gamma_r \frac{df}{f}
\]

If $p$ increases $B$ increases $f$ increases
II - Constant energy

\[ V_{p_B} = 0 \]

Beam debunches

\[ \frac{dp}{p} = \gamma^2 \frac{dR}{B} \]

\[ \frac{dp}{p} = \gamma \frac{df}{f} + \gamma^2 \frac{dR}{B} \]

If B increases

\[ R \text{ decreases} \]

\[ f \text{ increases} \]

III - Magnetic flat-top

\[ dB = 0 \]

Beam bunched with constant magnetic field

\[ \frac{dp}{p} = \gamma^2 \frac{dR}{B} = 0 \]

\[ \frac{dB}{B} = 0 = \gamma \frac{df}{f} \left[ 1 - \left( \frac{R^2}{\gamma^2} \right) \right] \frac{dp}{p} \]

\[ \frac{dB}{B} = 0 = \gamma \frac{df}{f} \left( \frac{\gamma^2 - R^2}{\gamma^2} \right) \frac{dR}{R} \]

If p increases

\[ R \text{ increases} \]

\[ f \text{ increases} \]

\[ \gamma < \gamma_e \]

\[ \gamma \text{ decreases} \]

IV - Constant frequency

\[ df = 0 \]

Beam driven by an external oscillator

\[ \frac{dp}{p} = \gamma^2 \frac{dR}{R} \]

\[ \frac{dR}{R} = \left[ 1 - \left( \frac{R^2}{\gamma^2} \right) \right] \frac{dp}{p} \]

\[ \frac{dR}{R} = \left( \gamma^2 - R^2 \right) \frac{dR}{R} \]

If p increases

\[ R \text{ increases} \]

\[ B \text{ decreases} \]

\[ \gamma < \gamma_e \]

\[ \gamma \text{ increases} \]

\[ \gamma_e \text{ decreases} \]

Four conditions - resume

<table>
<thead>
<tr>
<th>Beam</th>
<th>Parameter</th>
<th>Variations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Debunched</td>
<td>( \Delta p = 0 )</td>
<td>( B \uparrow, R \downarrow, f \uparrow )</td>
</tr>
<tr>
<td>Fixed orbit</td>
<td>( \Delta R = 0 )</td>
<td>( B \uparrow, p \uparrow, f \uparrow )</td>
</tr>
<tr>
<td>Magnetic flat-top</td>
<td>( \Delta B = 0 )</td>
<td>( p \uparrow, R \uparrow, f \uparrow (\eta &gt; 0) ) ( f \downarrow (\eta &lt; 0) )</td>
</tr>
<tr>
<td>External oscillator</td>
<td>( \Delta f = 0 )</td>
<td>( B \uparrow, p \downarrow, R \uparrow, f \downarrow (\eta &gt; 0) ) ( p \uparrow, R \uparrow, f \downarrow (\eta &lt; 0) )</td>
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</table>
Synchronous particle: particle that sees always the same phase (at each turn) in the RF cavity.

In order to keep the resonant condition, the particle must keep a constant energy. The phase of the synchronous particle must therefore be \( \phi_0 = 0 \) (circular machines convention).

Let's see what happens for a particle with the same energy and a different phase (e.g., \( \phi_1 \)).

\[
\Delta E = e \dot{V}_{RF} \sin \phi
\]

\( \Delta E \) represents the change in energy due to the RF field, \( \dot{V}_{RF} \) is the gradient of the RF field, and \( \phi \) is the phase of the particle.

Synchrotron oscillations:

1. The particle is accelerated.
2. Below transition, an increase in energy means an increase in revolution frequency.
3. The particle arrives earlier - tends toward \( \phi_0 \).

- The particle is decelerated.
- Decrease in energy - decrease in revolution frequency.
- The particle arrives later - tends toward \( \phi_1 \).

Phase space picture:

Phase space picture for particle motion. The trajectory shows the particle's evolution in phase and frequency space, highlighting the oscillatory behavior.

Synchronous particle with acceleration \( B \) increasing:

1. The phase of the synchronous particle is now \( \phi_s > 0 \) (circular machines convention).
2. The synchronous particle accelerates, and the magnetic field is increased accordingly to keep the constant radius \( R \).
3. The RF frequency is increased as well in order to keep the resonant condition.

\[
\Delta E = e \dot{V}_{RF} \sin \phi
\]

\[
\omega = \frac{e B}{\gamma m_0} = \frac{\omega_{RF}}{h}
\]

\( \omega \) is the angular frequency, \( eB \) is the gradient of the RF field, \( \gamma m_0 \) is the relativistic mass, and \( \omega_{RF} \) is the RF angular frequency.
Phase stability

\[ \phi = \omega_{AP} \cdot t \]

\[ V_{AP} \]

The symmetry of the case with \( B = \text{const} \) is lost.

Phase stability

\[ \phi_{s} < \phi < \pi - \phi_{s} \]

stable region

unstable region

separatrix

\[ \frac{\Delta \rho}{\rho} \]

RF acceleration for synchronous particle - energy gain

Let's assume a synchronous particle with a given \( \phi_{s} \geq 0 \)

We want to calculate its rate of acceleration, and the related rate of increase of \( B, f \).

\[ p = e B \rho \]

Want to keep \( \rho = \text{const} \)

\[ \frac{dp}{dt} = e \rho \frac{dB}{dt} = e \rho \dot{B} \]

Over one turn:

\[ (\Delta p)_{\text{syn}} = e \rho \dot{B} T_{\text{syn}} = e \rho B \frac{2\pi R}{\beta c} \]

We know that (relativistic equations):

\[ \Delta p = \frac{\Delta E}{\beta c} \]

\[ (\Delta E)_{\text{syn}} = e \rho B \frac{2\pi R}{\beta c} \]
On the other hand, for the synchronous particle:

\[
(\Delta E)_\text{turn} = e \rho \dot{B} \frac{2\pi R}{2\pi} \quad \text{for the synchronous particle:} \quad (\Delta E)_\text{turn} = e \dot{V}_s \sin \phi_s
\]

Therefore:

1. Knowing \( \phi_s \), one can calculate the increase rate of the magnetic field needed for a given RF voltage:

\[
\dot{B} = \frac{\dot{V}_s}{2\pi \rho R} \sin \phi_s
\]

2. Knowing the magnetic field variation and the RF voltage, one can calculate the value of the synchronous phase:

\[
\sin \phi_s = 2\pi \rho R \frac{\dot{B}}{\dot{V}_s} \quad \Rightarrow \quad \phi_s = \arcsin \left( 2\pi \rho R \frac{\dot{B}}{\dot{V}_s} \right)
\]

Example: PS

At the CERN Proton Synchrotron machine, one has:

\[
R = 100 \text{ m} \\
\dot{B} = 2.4 \text{ T/s}
\]

100 dipoles with \( L_{\text{eff}} = 4.398 \text{ m} \). The harmonic number is 20

Calculate:

1. The energy gain per turn
2. The minimum RF voltage needed
3. The RF Frequency when \( B = 1.23 \text{ T} \) (at extraction)

RF acceleration for non synchronous particle

Parameter definition (subscript "s" stands for synchronous particle):

- Revolution frequency: \( f = f_s + \Delta f \)
- RF phase: \( \phi = \phi_s + \Delta \phi \)
- Momentum: \( p = p_s + \Delta p \)
- Energy: \( E = E_s + \Delta E \)
- Azimuth angle: \( \theta = \theta_s + \Delta \theta \)

\[
\frac{\Delta s}{R} = \Delta \theta \\
\theta(t) = \int_0^t \omega(t) \, dt
\]
\[
\theta(t) = \int_0^t \omega(t) \, dt
\]

\[
\Delta \theta > 0 \implies \Delta \phi < 0
\]

Since \( f_{nf} = f_{rev} \)

\[
\Delta \phi = -\hbar \Delta \theta
\]

Over one turn \( \theta \) varies by \( 2\pi \)

\[
\phi \text{ varies by } \frac{2\pi}{h}
\]

1. Angular frequency

\[
\omega(t) = \frac{\partial}{\partial t} \left( \Delta \theta \right)
\]

\[
= -\frac{1}{\hbar} \frac{d}{dt} \left( \Delta \phi \right)
\]

\[
= -\frac{1}{\hbar} \frac{d}{dt} \left( \phi - \phi_0 \right)
\]

\[
= -\frac{1}{\hbar} \frac{d\phi}{dt} = 0 \text{ by definition}
\]

\[
\Delta \omega = -\frac{1}{\hbar} \frac{d\phi}{dt}
\]

2. Momentum

\[
\eta = \frac{\partial \omega}{\partial p} = \frac{\Delta \omega}{\Delta p}
\]

\[
\Delta p = \frac{p_i - \omega_i \eta}{\omega_i \eta} \left( -\frac{1}{\hbar} \frac{d\phi}{dt} \right)
\]

\[
\Delta p = -\frac{p_i}{\omega_i \eta} \frac{d\phi}{dt}
\]

3. Energy

\[
\frac{\partial E}{\partial p} = \nu
\]

\[
\Delta E = \nu = \omega R
\]

\[
\Delta E = -\frac{R p_i}{\eta h} \frac{d\phi}{dt}
\]

Derivation of equations of motion

Energy gain after the RF cavity

\[
\left( \Delta E \right)_{\text{gain}} = e \hat{V}_{\text{RF}} \sin \phi
\]

\[
\left( \Delta E \right)_{\text{gain}} = e \hat{V}_{\text{RF}} \sin \phi
\]

Average increase per time unit

\[
\frac{\left( \Delta E \right)_{\text{gain}}}{T_{\text{rev}}} = \frac{e}{2\pi} \hat{V}_{\text{RF}} \sin \phi
\]

\[
2\pi R \hat{p} = e \hat{V}_{\text{RF}} \sin \phi \quad \text{valid for any particle!}
\]

\[
2\pi \left( R \hat{p} - R_i \hat{p}_i \right) = e \hat{V}_{\text{RF}} \left( \sin \phi - \sin \phi_i \right)
\]

\[
\Delta \omega = -\frac{1}{\hbar} \frac{d\phi}{dt}
\]
After some development (see J. Le Duff, in Proceedings CAS 1992, CERN 94-01)

\[ 2\pi \frac{d}{dt} \left( \frac{\Delta \phi}{\omega_{\phi}} \right) = e \dot{V}_{x}\phi (\sin \phi - \sin \phi_s) \]

An approximated version of the above is

\[ \frac{d\Delta \phi}{dt} = \frac{e \dot{V}_{x}\phi}{2\pi R_s} (\sin \phi - \sin \phi_s) \]

Which, together with the previously found equation:

\[ \frac{d\phi}{dt} = -\omega_{\phi} \frac{\eta h}{P_s} \Delta p \]

Describes the motion of the non-synchronous particle in the longitudinal phase space \((\Delta \phi, \phi)\).

1. First approximation - combining the two equations:

We assume that \(A\) and \(B\) change very slowly compared to the variable \(\Delta \phi = \phi - \phi_s\).

\[ \frac{d}{dt} \left( \frac{1}{B} \frac{d\phi}{dt} \right) = A (\sin \phi - \sin \phi_s) = 0 \]

We can also define:

\[ \Omega_0^2 = \frac{\omega_{\phi}^2}{\cos \phi_s} = \frac{e \dot{V}_{x}\phi \eta h c^2}{2\pi R_s^2 E_s} \]

with \(\Omega_0^2 = -AB\) and \(\Omega_{\phi}^2 = 2\Omega_0^2 \cos \phi_s\).

\[ \frac{d^2\phi}{dt^2} + \Omega_{\phi}^2 (\sin \phi - \sin \phi_s) = 0 \]

2. Second approximation

\[ \sin \phi = \sin(\phi_s + \Delta \phi) \]

\[ = \sin \phi_s \cos \Delta \phi + \cos \phi_s \sin \Delta \phi \]

\[ \Delta \phi \text{ small} \Rightarrow \sin \phi = \sin \phi_s + \cos \phi_s \Delta \phi \]

\[ \frac{d\phi_s}{dt} = 0 \Rightarrow \frac{d^2\phi}{dt^2} = \frac{d^2\phi_s}{dt^2} (\phi_s + \Delta \phi) + \frac{d^2\Delta \phi}{dt^2} \]

by definition

\[ \frac{d^2\Delta \phi}{dt^2} = \Omega_{\phi}^2 \Delta \phi = 0 \]

Harmonic oscillator.
Stability condition for $\phi$

Stability is obtained when the angular frequency of the oscillator, $\Omega_s^2$, is real positive:

$$\Omega_s^2 = \frac{e V_F \eta h \sqrt{c^2}}{2 \pi R_s^2 E_s} \cos \phi_s \quad \Rightarrow \quad \Omega_s^2 > 0 \quad \Leftrightarrow \quad \eta \cos \phi_s > 0$$

Stable in the region if

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\pm\frac{1}{2}\pi$</th>
<th>$\pm\frac{1}{2}\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta &gt; 0$</td>
<td>$\eta &lt; 0$</td>
<td>$\eta &lt; 0$</td>
</tr>
</tbody>
</table>

acceleration  deceleration

Small amplitude oscillations - orbits

For $\eta \cos \phi_s > 0$, the motion around the synchronous particle is a stable oscillation:

$$\begin{align*}
\Delta \phi &= \Delta \phi_{\text{max}} \sin (\Omega_s t + \phi_s) \\
\Delta P &= \Delta P_{\text{max}} \cos (\Omega_s t + \phi_s)
\end{align*}$$

with $\Delta P_{\text{max}} = \frac{\Omega_s}{B} \Delta \phi_{\text{max}}$

Lepton machines $e^+, e^-$

$\beta = 1$, $\gamma$ large, $\eta = -\alpha_p$

$$\omega_s = \frac{c}{R_s}, \quad p_s = \frac{E_s}{c} \quad \Rightarrow \quad \Omega_s = \frac{c}{R_s} \left[ \frac{e V_F \alpha_s h}{2 \pi E_s} \cos \phi_s \right]^{1/2}$$

Number of synchrotron oscillations per turn:

$$Q_s = \frac{\Omega_s}{\omega_s} = \left[ \frac{e V_F \alpha_s h}{2 \pi E_s} \cos \phi_s \right]^{1/2} \quad \text{"synchrotron tune"}$$

N.B.: in these machines, the RF frequency does not change

Large amplitude oscillations

$$\ddot{\phi} + \Omega_s^2 \cos \phi \left( \sin \phi - \sin \phi_s \right) = 0$$

Multiplying by $\dot{\phi}$ and integrating

$$\left( \frac{\dot{\phi}}{2} \frac{\Omega_s^2}{\cos \phi} \right) \left( \cos \phi + \phi \sin \phi_s \right) = \text{cte}$$

Constant of motion

Equation of the separatrix

$$\frac{\dot{\phi}^2}{2} \frac{\Omega_s^2}{\cos \phi} \left( \cos \phi + \phi \sin \phi_s \right) = - \frac{\Omega_s^2}{\cos \phi_s} \left( \cos (\pi - \phi_s) + (\pi - \phi_s) \sin \phi_s \right)$$
Energy diagram

If the total energy is above this limit, the motion is unbounded.

Phase space trajectories

Phase space trajectories for different synchronous phases